V.5 Symmetric operators and the Cayley transform

Definition. Let S be a symmetric (not necessarily densely defined) operator on H. Denote by C_S the operator

$$C_S = (S - iI)(S + iI)^{-1}.$$

Then C_S is an operator on H, which is called the **Cayley transform of the operator** S.

Theorem 32 (properties of C_S). *Let S be a symmetric operator on H and let C_S be its Cayley transform. Then*

(a) *C_S is a linear isometry of $D(C_S) = R(S + iI)$ onto $R(C_S) = R(S - iI)$.*
(b) *$I - C_S = 2i(S + iI)^{-1}$; in particular, the operator $I - C_S$ is one-to-one and $R(I - C_S) = D(S)$.*
(c) *$S = i(I + C_S)(I - C_S)^{-1}$.*
(d) *C_S is closed $\Leftrightarrow$ S is closed $\Leftrightarrow$ $D(C_S)$ is closed $\Leftrightarrow$ $R(C_S)$ is closed.*

Lemma 33 (on an isometric operator). *Let U be any operator on H, which is an isometry of $D(U)$ onto $R(U)$. Then*

(a) *$\langle Ux, Uy\rangle = \langle x, y\rangle$ for any $x, y \in D(U)$. In particular: U is unitary if and only if $D(U) = R(U) = H$.*
(b) $\mathrm{Ker}(I - U) = D(U) \cap (R(I - U))^\perp$. *In particular, if $R(I - U)$ is dense in H, then $I - U$ is one-to-one.*

Theorem 34 (range of the Cayley transform). *Let U be an operator on H, which is an isometry of $D(U)$ onto $R(U)$. Suppose that $I-U$ is one-to-one. Then the operator $S = i(I+U)(I-U)^{-1}$ is symmetric and $C_S = U$. Further, S is densely defined if and only if $R(I - U)$ is dense.*

Theorem 35 (Cayley transform for selfadjoint operators).

(a) *Let S be a symmetric operator on H. Then S is selfadjoint if and only if C_S is a unitary operator.*
(b) *Let U be a unitary operator na H such that $I - U$ is one-to-one. Then the operator $S = i(I + U)(I - U)^{-1}$ is selfadjoint and $C_S = U$.*

Remarks.

(1) Let S and T be symmetric operators on H. Then $S \subset T$ if and only if $C_S \subset C_T$.
(2) Let S be a densely defined closed symmetric operator on H. The codimensions of the subspaces $D(C_S)$ and $R(C_S)$ (i.e., the dimensions of their orthogonal complements) are called the **deficiency indices** of the operator S. Then:
  - S is selfadjoint if and only if both deficiency indices are zero.
  - S is a maximal symmetric operator if and only if at least one of the deficiency indices is zero.
  - S has a selfadjoint extension if and only if both deficiency indices are the same (i.e., if and only if there exists a linear isometry of $(D(C_S))^\perp$ onto $(R(C_S))^\perp$).

(3) Let S be a closed symmetric operator on H, not necessarily densely defined. Also in this case one may define the deficiency indices. Moreover:
  - If $D(C_S) = H$ or $R(C_S) = H$, then S is densely defined.

  Hence, also in this case the first of the above equivalences holds. It easily follows that in the second assertion $\Leftarrow$ holds and in the third assertion $\Rightarrow$ holds. The validity of the converse implications seems not to be clear.