VI. Spectral measures and spectral decompositions

Convention: In this chapter we will consider bounded and unbounded operators on complex Hilbert spaces. Hence, Hilbert spaces are assumed to be complex, except for the very last section.

VI.1 Measurable calculus and spectral measure for normal bounded operators

Proposition 1 (Lax-Milgram). *Let H be a Hilbert space and $B : H \times H \to \mathbb{C}$ a mapping satisfying the following properties.*

- *$x \mapsto B(x, y)$ is linear for each $y \in H$.*
- *$y \mapsto B(x, y)$ is conjugate linear for each $x \in H$.*
- *$\|B\| = \sup\{|B(x, y)| ; x, y \in B_H\} < \infty$.*

Then there is a unique $T \in L(H)$ such that $B(x, y) = \langle Tx, y\rangle$ for $x, y \in H$. Moreover, $\|T\| = \|B\|$.

Constructing the spectral measure of a normal operator - Step 1. Let H be a Hilbert space and let $T \in L(H)$ be a normal operator. Let $f \mapsto \tilde{f}(T)$, $f \in \mathcal{C}(\sigma(T))$, be the continuous functional calculus for T. For any $x, y \in H$ let $E_{x,y}$ denote the (unique) complex Radon measure on $\sigma(T)$ satisfying

$$\left\langle \tilde{f}(T)x, y \right\rangle = \int_{\sigma(T)} f \,\mathrm{d}E_{x,y}, \qquad f \in \mathcal{C}(\sigma(T)).$$

Proposition 2 (properties of the measures $E_{x,y}$). *Using the above notation, the following holds:*

(a) *$x \mapsto E_{x,y}$ is linear for each $y \in H$.*
(b) *$y \mapsto E_{x,y}$ is conjugate linear for each $x \in H$.*
(c) *$E_{x,x}$ is a non-negative measure for each $x \in H$.*
(d) *$\|E_{x,y}\| \le \|x\| \cdot \|y\|$ for $x, y \in H$.*
(e) *$E_{x,y} = \frac{1}{4}(E_{x+y,x+y} - E_{x-y,x-y} + iE_{x+iy,x+iy} - iE_{x-iy,x-iy})$ for $x, y \in H$.*

Measurable calculus and the spectral measure. We use the above notation.

- Denote by $\mathcal{A}$ the σ-algebra of all the subsets of $\sigma(T)$ which are $E_{x,y}$-measurable for each $x, y \in H$. (Recall that A is $E_{x,y}$-measurable if and only if there are Borel sets B, C such that $B \subset A \subset C$ and $|E_{x,y}|(B \setminus C) = 0$.) Then $\mathcal{A}$ is the σ-algebra of all the subsets of $\sigma(T)$ which are $E_{x,x}$-measurable for each $x \in H$.
- Let $f : \sigma(T) \to \mathbb{C}$ be a bounded $\mathcal{A}$-measurable function By $\tilde{f}(T)$ we denote the operator in $L(H)$ satisfying

$$\left\langle \tilde{f}(T)x, y \right\rangle = \int_{\sigma(T)} f \,\mathrm{d}E_{x,y}, \qquad x, y \in H.$$

  Its existence and uniqueness is provided by Proposition 1. The assignement $f \mapsto \tilde{f}(T)$ is called the **measurable calculus** for T.
- For $A \in \mathcal{A}$ set $E_T(A) = \widetilde{\chi_A}(T)$. The assignement $E_T : A \mapsto E_T(A)$ is called the **spectral measure** of T.
- Denote by $\mathcal{N}$ the subfamily of $\mathcal{A}$ formed by the sets which are $|E_{x,y}|$-null for each $x, y \in H$. Then $\mathcal{N}$ is the family of all the sets which are $E_{x,x}$-null for each $x \in H$.

- Denote by $L^\infty(E_T)$ the space of all the bounded $\mathcal{A}$-measurable functions on $\sigma(T)$, where we identify the functions which are equal everywhere except on a set from $\mathcal{N}$. Equip $L^\infty(E_T)$ with the norm

$$\|f\| = \operatorname*{ess\,sup}_{\lambda\in\sigma(T)} |f(\lambda)| = \inf\{c > 0; \{\lambda \in \sigma(T); f(\lambda) > c\} \in \mathcal{N}\}.$$

  Then $L^\infty(E_T)$ is a commutative C^*-algebra (with the pointwise multiplication and the involution defined as the complex conjugation).
- $\tilde{f}(T)$ is defined exactly for $f \in L^\infty(E_T)$. Moreover, $\tilde{f}(T)$ is then well defined, i.e., $\tilde{f}(T) = \tilde{g}(T)$ whenever $f = g$ except on a set from $\mathcal{N}$.

Lemma 3 (a consequence of Luzin's theorem).

(a) *Let K be a compact metric space and let μ be a non-negative finite Borel measure on K. Let $f : K \to \mathbb{C}$ be a bounded μ-measurable function. Then there is a uniformly bounded sequence (f_n) in $\mathcal{C}(K)$ such that $f_n \to f$ μ-almost everywhere. In particular, there is a bounded Borel function g on $\sigma(T)$ such that $f = g$ μ-almost everywhere.*

(g) *Let H be a separable Hilbert space and let $T \in L(H)$ be a normal operator. Let $f \in L^\infty(E_T)$ Then there is a uniformly bounded sequence (f_n) in $\mathcal{C}(\sigma(T))$ such that $f_n \to f$ except on a set from $\mathcal{N}$. In particular, there exists a bounded Borel function g on $\sigma(T)$ such that $f = g$ except on a set form $\mathcal{N}$.*

Theorem 4 (properties of the measurable calculus). *Let H be a Hilbert space and $T \in L(H)$ be a normal operator.*

(a) *$f \mapsto \tilde{f}(T)$ is an isometric $*$-isomorphism of $L^\infty(E)$ into $L(H)$.*

(b) *If (f_n) is a bounded sequence in $L^\infty(E)$ which pointwise converges to a function f (except on a set from $\mathcal{N}$), then $f \in L^\infty(E)$ and, moreover,*

$$\left\langle \tilde{f}_n(T)x, y \right\rangle \to \left\langle \tilde{f}(T)x, y \right\rangle, \qquad x, y \in H.$$

(c) *$\sigma(\tilde{f}(T)) = \text{ess rng}(f) = \{\lambda \in \mathbb{C}; \forall r > 0 : f^{-1}(U(\lambda, r)) \notin \mathcal{N}\}$ for $f \in L^\infty(E)$.*

(d) *$\tilde{f}(T)$ is a normal operator for each $f \in L^\infty(E)$. $\tilde{f}(T)$ is self-adjoint if and only if f is essentially real-valued (i.e., $f(\lambda) \in \mathbb{R}$ except on a set from $\mathcal{N}$).*

(e) *$\tilde{g}(\tilde{f}(T)) = \widetilde{g \circ f}(T)$ whenever $f \in L^\infty(E)$ and g is continuous on $\sigma(\tilde{f}(T))$ (see (c)).*

(f) *If $S \in L(H)$ commutes with T, then S commutes with $\tilde{f}(T)$ for each $f \in L^\infty(E)$.*