VI.4 Normal unbounded operators

Definition. A densely defined closed operator T on a Hilbert space is called **normal** if $T^*T = TT^*$.

Lemma 19 (on T^*T). *Let T be a closed densely defined operator on H. Then:*

(a) *$I + T^*T$ is a bijection of $D(T^*T)$ onto H.*
(b) *Denote by B the inverse operator to $I + T^*T$ and $C = TB$. Then B and C belong to $L(H)$ and their norms are at most 1. Moreover, B is positive.*
(c) *T^*T is selfadjoint and T is the closure of $T|_{D(T^*T)}$.*

Lemma 20. *Let T be a normal operator on H. Then:*

(a) *$D(T) = D(T^*)$*
(b) *$\|Tx\| = \|T^*x\|$ for $x \in D(T)$.*
(c) *If $S \supset T$ is normal, then $S = T$.*

Theorem 21 (spectral decomposition of an unbounded normal operator). *If T is a normal operator on H, then there exists a unique abstract spectral measure E in H such that $T = \int \mathrm{id}\, \mathrm{d}E$. This measure can be described as follows: Let B be the operator from Lemma 19. For $j \in \mathbb{N}$ let $P_j = \widetilde{\chi_{(\frac{1}{j+1},\frac{1}{j}]}}(B)$. Then TP_j is a bounded normal operator, let E^j be its spectral measure and let $\mathcal{A}_j$ be the corresponding σ-algebra. Then the sought measure E is given by*

$$E(A)x = \sum_{j=1}^{\infty} E^j(A)P_jx, \quad x \in H, A \in \mathcal{A} = \bigcap_{j\in\mathbb{N}} \mathcal{A}_j.$$

Corollary 22. *Let T be a normal operator on H. Then T is bounded if and only if $\sigma(T)$ is a bounded set.*

Corollary 23. *Let E be an abstract spectral measure in a Hilbert space H defined on a σ-algebra $\mathcal{A}$. Let $f : \mathbb{C} \to \mathbb{C}$ be a $\mathcal{A}$-measurable function and let $T = \int f\, \mathrm{d}E$. Then T is a normal operator and its spectral measure (i.e., the measure from Theorem 21) is the image of E under f (in the sense of Lemma 16).*