# FUNCTIONAL ANALYSIS 2

SUMMER SEMESTER 2021/2022

PROBLEMS TO CHAPTER V

## Problems to Section V.1 – special types of operators

**Problem 1.** Let $H$ be a Hilbert space and let $U \in L(H)$ be a partial isometry. By definition it means that there is a closed subspace $Y \subset H$ such that $U|_Y$ is an isometry and $U|_{Y^\perp} = 0$. Denote by $Z$ the range of $U$.

(1) Show that $\langle Ux, Uy \rangle = \langle x, y \rangle$ for $x, y \in Y$.
(2) Show that $U^*$ is also a partial isometry ($U^*|_Z$ is an isometry and $U^*_{Z^\perp} = 0$).
(3) Show that $U^*U$ is the orthogonal projection onto $Y$.
(4) Show that $UU^*$ is the orthogonal projection onto $Z$.

**Problem 2.** Let $H$ be a Hilbert space and let $U \in L(H)$.

(1) Assume that $U^*U$ is an orthogonal projection. Show that $U$ is a partial isometry.
(2) Assume that $UU^*$ is an orthogonal projection. Show that $U$ is a partial isometry.
(3) Assume that $UU^*U = U$. Show that $U$ is a partial isometry.

**Problem 3.** Let $X$ be a (complex) Banach space and let $T \in L(X)$.

(1) Show that $\overline{\sigma_p(T)} \subset \sigma_{ap}(T)$.
(2) Show that $\lambda \in \mathbb{C} \setminus \sigma_{ap}(T)$ if and only if the operator $\lambda I - T$ is bounded from below (i.e., the function $x \mapsto \|(\lambda I - T)x\|$ is bounded from below on the unit sphere).
(3) Show that operators which are bounded below form an open set in $L(X)$.
(4) Deduce that $\sigma_{ap}(T)$ is a closed set (and provide an alternative proof of (1)).
(5) Show that $\partial\sigma(T) \subset \sigma_{ap}(T)$.

`Hint:` *(5) Let $\lambda \in \partial\sigma(T)$. Then there is a sequence $(\lambda_n)$ in $\rho(T)$ converging to $\lambda$. By Theorem IV.7(3) we get $\|(\lambda_n I - T)^{-1}\| \to \infty$, hence there are $x_n \in S_X$ with $\|(\lambda_n I - T)^{-1} x_n\| \to \infty$. Show that $y_n = \frac{(\lambda_n I - T)^{-1} x_n}{\|(\lambda_n I - T)^{-1} x_n\|}$ witness that $\lambda \in \sigma_{ap}(T)$.*

**Problem 4.** Let $H$ be a (complex) Hilbert space and let $T \in L(H)$.

(1) Show that $\sigma(T^*) = \{\overline{\lambda}; \lambda \in \sigma(T)\}$.
(2) Show that $\lambda \in \sigma_p(T)$ if and only if the range of $\overline{\lambda} I - T$ is not dense.
(3) Show that $\lambda \in \sigma_r(T) \Rightarrow \overline{\lambda} \in \sigma_p(T^*)$.
(4) Show that $\lambda \in \sigma_p(T) \Rightarrow \overline{\lambda} \in \sigma_r(T^*) \cup \sigma_p(T^*)$.

`Hint:` *(1) See Proposition IV.26(d). (2) Recall that $\ker T = (R(T^*))^\perp$.*

**Problem 5.** Let $H = \ell_2(\mathbb{N})$ (the complex version) and let $S : H \to H$ be defined by

$$S(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots).$$

(1) Compute $S^*$, $S^*S$ and $SS^*$ and show that $S$ is a partial isometry.
(2) Show that $\sigma_p(S) = U(0,1)$ (the open unit disc) and deduce that $\sigma(S) = \sigma_{ap}(S) = \overline{U(0,1)}$.
(3) Deduce that $\sigma(S^*) = \overline{U(0,1)}$.
(4) Show that $\sigma_p(S^*) = \emptyset$.
(5) Deduce that $\sigma_r(S) = \emptyset$, $\sigma_r(S^*) = U(0,1)$ and $\sigma_c(S) = \sigma_c(S^*) = \{\lambda \in \mathbb{C}; |\lambda| = 1\}$.
(6) Show that for $\lambda \in U(0,1)$ we have $\|(\lambda I - S^*)x\| \geq (1 - |\lambda|)\,\|x\|$.
(7) Deduce that $\sigma_{ap}(S^*) = \{\lambda \in \mathbb{C}; |\lambda| = 1\}$.
(8) Given a complex unit $\lambda$, find a sequence $(x_n)$ in $H$ witnessing that $\lambda \in \sigma_{ap}(S^*)$.

`Hint:` *(3) Use Problem 4(1). (5) Use Problem 4(3,4). (8) Take the normalization of vectors* $(\lambda^n, \dots, \lambda, 1, 0, 0, \dots)$.

**Problem 6.** Let $H = \ell_2(\mathbb{Z})$ (the complex version) and let $S : H \to H$ be defined by

$$S((x_n)) = (x_{n-1}).$$

(1) Compute $S^*$, $S^*S$ and $SS^*$ and show that $S$ is a unitary operator.
(2) Show that $\sigma_p(S) = \emptyset$.
(3) Show that $\sigma(S) = \sigma_{ap}(S) = \sigma_c(S) = \{\lambda \in \mathbb{C}; |\lambda| = 1\}$.
(4) Given a complex unit $\lambda$, find a sequence $(x_n)$ in $H$ witnessing that $\lambda \in \sigma_{ap}(S)$.
(5) Solve (2)–(4) for $S^*$.

`Hint:` *(3) Since* $S^* = S^{-1}$*, it follows from (2) that* $\sigma_p(S^*) = \emptyset$ *as well. Use Problem 4(3). (4) Modify the vectors from Problem 5(8).*

**Problem 7.** Let $H$ be a (complex) Hilbert space and $T \in L(H)$. Show that $W(T^*) = \{\overline{\lambda}; \lambda \in W(T)\}$ and deduce that $w(T^*) = w(T)$.

**Problem 8.** (1) Let $H = \mathbb{R}^2$ be the two-dimensional real Hilbert space. Let $T \in L(H)$ be defined by $T(x_1, x_2) = (-x_2, x_1)$. Show that $W(T) = \{0\}$ and $w(T) = 0$.
(2) Let $H$ be a real Hilbert space whose dimension is an even natural number. Find a unitary operator $T \in L(H)$ with $w(T) = 0$.
(3) Let $H$ be an infinite dimensional real Hilbert space. Find a unitary operator $T \in L(H)$ with $w(T) = 0$.
(4) Let $H$ be a real Hilbert space of a finite odd dimension. Let $T \in L(H)$ be an invertible operator. Show that $w(T) > 0$.

`Hint:` *(4) A square matrix of odd order with real entries has at least one real eigenvalue.*

**Problem 9.** Let $H = \mathbb{C}^2$ be the two-dimensional complex Hilbert space. Let $T \in L(H)$ be defined by $T(x_1, x_2) = (-x_2, x_1)$. Show that $W(T) = \{it; t \in [-1, 1]\}$ and deduce that $w(T) = 1$.

**Problem 10.** Let $H = \mathbb{C}^2$ be the two-dimensional complex Hilbert space. Let $T \in L(H)$ be defined by $T(x_1, x_2) = (0, x_1)$. Show that $W(T) = \overline{U(0, \frac{1}{2})}$. Deduce that $w(T) = \frac{1}{2}$ and hence the constant $\frac{1}{2}$ in Proposition V.4(a) is optimal.

**Problem 11.** Let $H = \mathbb{C}^3$ be the three-dimensional complex Hilbert space. Let $T \in L(H)$ be defined by $T(x_1, x_2, x_3) = (0, x_1, x_2)$. Compute $W(T)$ and $w(T)$.

Hint: *Find maximum of the function $a_1a_2 + a_2a_3$ over $\{(a_1, a_2, a_3) \in \mathbb{R}^3; a_1^2 + a_2^2 + a_3^2 = 1\}$ (for example using Lagrange multipliers).*

**Problem 12.** (1) Let $S$ be the operator from Problem 5. Show that $W(S) = U(0, 1)$.
(2) Let $S$ be the operator from Problem 6. Show that $W(S) = U(0, 1)$.

Hint: *To prove '$\subset$' use the Cauchy-Schwarz inequality, including the criterion for equality. To prove '$\supset$' in (1) use the eigenvectors of $S$, in (2) use the same vectors completed by zeros.*

**Problem 13.** (1) Find a polarization formula (i.e., an analogue of Lemma V.3) for self-adjoint operators on a real Hilbert space.
(2) Show that the formula is valid only for self-adjoint operators.

**Problem 14.** Let $H = \ell^2(\mathbb{N})$ (the complex version) and let $(\alpha_n)$ be a bounded sequence of complex numbers. For $\boldsymbol{x} = (x_n) \in H$ let $T(\boldsymbol{x}) = (\alpha_n x_n)$.

(1) Show that $T \in L(H)$, $\|T\| = \sup_n |\alpha_n|$, determine $\sigma(T)$ and $\sigma_p(T)$.
(2) Compute $T^*$ and show that $T$ is normal.
(3) Compute $|T|$ and find the polar decomposition of $T$.
(4) Let $S$ be the operator from Problem 5. Let $T_1 = ST$ and $T_2 = S^*T$. Compute $T_1^*$ and $T_2^*$. Are these operators normal?
(5) Compute $|T_1|$, $|T_1^*|$, $|T_2|$, $|T_2^*|$.
(6) Compute the polar decompositions of operators $T_1, T_1^*, T_2, T_2^*$.

**Problem 15.** Let $H$ be a (complex) Hilbert space and let $T \in L(H)$ be a compact normal operator. Consider the formula for $T$ from Theorem V.9.

(1) Show that $\lambda_k$ are exactly nonzero eigenvalues of $T$.
(2) Show that $T$ is self-adjoint if and only if all $\lambda_k$ are real.
(3) Show that $T$ is positive if and only if all $\lambda_k$ are positive.
(4) Find the representation of $T$ given by Theorem V.11.

Hint: *(4) Take $\alpha_k = |\lambda_k|$, $e_k = x_k$ and $f_k$ a suitable multiple of $x_k$.*

**Problem 16.** Let $H$ be a (complex) Hilbert space and let $T \in L(H)$ be a compact operator. Consider the formula for $T$ from Theorem V.11.

(1) Find a similar formula for $T^*$.
(2) Find a similar formula for $|T|$ and $|T^*|$.
(3) Compute the polar decompositions of $T$ and $T^*$.

## Problems to Section V.2 – unbounded operators

**Problem 17.** Let $T$ be a densely defined closed operator from a Banach space $X$ to a Banach space $Y$. Show that $T$ is everywhere defined if and only if $T$ is continuous.

**Problem 18.** Let $X = \ell^p$ where $p \in [1, \infty)$ or $X = c_0$. Let $\boldsymbol{z} = (z_n)$ be a sequence of (real or complex) numbers. Let

$$D(M_{\boldsymbol{z}}) = \{(x_n) \in X; (x_n z_n) \in X\}$$

and define the operator $\mathcal{M}_{\boldsymbol{z}}$ by

$$M_{\boldsymbol{z}}((x_n)) = (x_n z_n), \quad (x_n) \in D(M_{\boldsymbol{z}}).$$

(1) Show that $M_{\boldsymbol{z}}$ is a densely defined closed operator on $X$.
(2) Show that $M_{\boldsymbol{z}}$ is bounded (hence everywhere defined) if and only if the sequence $\boldsymbol{z}$ is bounded. Show that in this case $\|M_{\boldsymbol{z}}\| = \|\boldsymbol{z}\|_\infty$.

**Problem 19.** Let $X = \ell^\infty$ and define the operator $M_{\boldsymbol{z}}$ in the same way as in Problem 18.

(1) Show that $M_{\boldsymbol{z}}$ is bounded if and only if the sequence $\boldsymbol{z}$ is bounded. Show that in this case $M_{\boldsymbol{z}}$ is everywhere defined and $\|M_{\boldsymbol{z}}\| = \|\boldsymbol{z}\|_\infty$.
(2) Show that $M_{\boldsymbol{z}}$ is a closed operator.
(3) Show that $M_{\boldsymbol{z}}$ is not densely defined unless $\boldsymbol{z}$ it bounded.

**Problem 20.** Let $(\Omega, \Sigma, \mu)$ be a measure space with $\mu$ semifinite (i.e., whenever $A \in \Sigma$ is such that $\mu(A) > 0$, then there is $B \in \Sigma$ such that $B \subset A$ and $0 < \mu(B) < \infty$). Let $X = L^p(\mu)$ where $p \in [1, \infty)$. Let $g$ be a measurable function on $\Omega$. Set

$$D(M_g) = \{f \in X; fg \in X\}$$

and define the operator $M_g$ by

$$M_g(f) = fg, \quad f \in D(M_g).$$

(1) Show that $M_g$ is a densely defined closed operator on $X$.
(2) Show that $M_g$ is bounded (hence everywhere defined) if and only if the function $g$ is essentially bounded. Show that in this case $\|M_g\| = \|g\|_\infty$.

**Problem 21.** Let $X = L^\infty((0,1))$ and define the operator $M_g$ in the same way as in Problem 20.

(1) Show that $M_g$ is bounded if and only if the function $g$ is essentially bounded. Show that in this case $M_g$ is everywhere defined and $\|M_g\| = \|g\|_\infty$.
(2) Show that $M_g$ is a closed operator.
(3) Show that $M_g$ is not densely defined unless $g$ it essentially bounded.

**Problem 22.** Let $X = \ell^p$, where $p \in (1, \infty)$. Let

$$Y = \{(x_n) \in X; (nx_n) \in X \ \& \ \sum_{n=1}^{\infty} x_n = 0\}.$$

(1) Show that $(nx_n) \in X$ implies $(x_n) \in \ell^1$ and deduce that $Y$ is a well-defined linear subspace of $X$.
(2) Show that $Y$ is dense in $X$.
(3) Define the operator $T$ by $T((x_n)) = (nx_n)$ for $(x_n) \in D(T) = Y$. Show that $T$ is a closed operator.

`Hint:` *(1) Use the Hölder inequality. (2) Approximate any finitely supported vector by an element of $Y$. (3) Use the definitions and the Hölder inequality.*

**Problem 23.** Let $X = L^p((1, \infty))$, where $p \in (1, \infty)$. Let $\varphi(t) = t$ for $t \in (1, \infty)$ and

$$Y = \{f \in X; \varphi \cdot f \in X \ \& \ \int_1^\infty f = 0\}.$$

(1) Show that $\varphi \cdot f \in X$ implies $f \in L^1((1, \infty))$ and deduce that $Y$ is a well-defined linear subspace of $X$.
(2) Show that $Y$ is dense in $X$.
(3) Define the operator $T$ by $T(f) = \varphi \cdot f$ for $(x_n) \in D(T) = Y$. Show that $T$ is a closed operator.

`Hint:` *(1) Use the Hölder inequality. (2) Approximate characteristic functions of bounded measurable sets by elements of $Y$ and use density of simple integrable functions in $X$. (3) Use the definitions and the Hölder inequality.*

**Problem 24.** Let $X = L^p((1,\infty))$, where $p \in (1,\infty)$. Let $\varphi(t) = [t]$ (the integer part of $t$) for $t \in (1,\infty)$ and

$$Y = \{f \in X; \varphi \cdot f \in X \ \& \ \int_1^\infty f = 0\}.$$

(1) Show that $\varphi \cdot f \in X$ implies $f \in L^1((1,\infty))$ and deduce that $Y$ is a well-defined linear subspace of $X$.
(2) Show that $Y$ is dense in $X$.
(3) Define the operator $T$ by $T(f) = \varphi \cdot f$ for $(x_n) \in D(T) = Y$. Show that $T$ is a closed operator.

`Hint:` *Proceed similarly as in Problem 23.*

**Problem 25.** Let $X = L^p((0,1))$ where $p \in [1,\infty)$. Set

$$Y = \{f \in AC([0,1]); f' \in X\},$$

where by $AC([0,1])$ we denote the space of functions which are absolutely continuous on $[0,1]$. Define operators $T_j$, $j = 1,\dots,6$, all of them by the same formula $T_j(f) = f'$, with domains

$$\begin{aligned} D(T_1) &= Y, & D(T_4) &= \{f \in Y; f(0) = f(1) = 0\},\\ D(T_2) &= \{f \in Y; f(0) = 0\}, & D(T_5) &= \{f \in Y; f(0) = f(1)\},\\ D(T_3) &= \{f \in Y; f(1) = 0\}, & D(T_6) &= \{f \in Y; f(0) = -f(1)\}. \end{aligned}$$

Show that all these operators are densely defined and closed (consider $Y$ as a subspace of $X$).

`Hint:` *To prove they are densely defined use the density of test functions in $L^p((0,1))$. To prove they are closed use the boundedness of the operator $V : X \to X$ defined by $V(f)(t) = \int_0^t f$, $t \in (0,1)$, $f \in X$.*

**Problem 26.** Let $X = L^p((0,\infty))$ where $p \in [1,\infty)$. Set

$$Y = \{f \in AC_{loc}([0,\infty)); f \in X \ \& \ f' \in X\},$$

where by $AC_{loc}([0,\infty))$ we denote the space of functions defined on $[0,\infty)$ which are absolutely continuous on $[0,R]$ for each $R \in (0,\infty)$ (considered as a subspace of $X$). Define operators $T_j$, $j = 1,2$, all of them by the same formula $T_j(f) = f'$, with domains

$$D(T_1) = Y, \qquad D(T_2) = \{f \in Y; f(0) = 0\}.$$

(1) Show that $\lim_{t\to\infty} f(t) = 0$ for each $f \in Y$.
(2) Show that $T_1$ and $T_2$ are densely defined.
(3) Show that $T_1$ and $T_2$ are closed.

`Hint:` *(1) Let $f \in Y$. Consider the function $g = |f|^p$. Compute $g'$ and show that $g' \in L^1((0,\infty))$ (using the Hölder inequality). Deduce that $g$ has a finite limit in $+\infty$ and that this limit has to be zero as $g \in L^1((0,\infty))$. (2) Use the density of test functions in $L^p((0,\infty))$. (3) Use the analogue of the operator $V$ from the hint to Problem 25 on intervals $(0,R)$ for $R \in (0,\infty)$.*

**Problem 27.** Let $X = L^p(\mathbb{R})$ where $p \in [1, \infty)$. Set

$$Y = \{f \in AC_{loc}(\mathbb{R}); f \in X \ \&\ f' \in X\},$$

where by $AC_{loc}(\mathbb{R})$ we denote the space of functions defined on $\mathbb{R}$ which are absolutely continuous on $[-R, R]$ for each $R \in (0, \infty)$ (considered as a subspace of $X$). Fix a closed set $F \subset \mathbb{R}$ of Lebesgue measure zero. Define operators $T_j$, $j = 1, 2, 3$, all of them by the same formula $T_j(f) = f'$, with domains

$$D(T_1) = Y, \qquad D(T_2) = \{f \in Y; f(0) = 0\}, \qquad D(T_3) = \{f \in Y; f|_F = 0\}.$$

(1) Show that $\lim\limits_{t \to \pm\infty} f(t) = 0$ for each $f \in Y$.
(2) Show that $T_1, T_2$ and $T_3$ are densely defined.
(3) Show that $T_1, T_2$ and $T_3$ are closed.

`Hint:` *(1) Use Problem 26. (2) Using density of test functions in $L^p(J)$ for any open interval $J$ and the assumption that $F$ has measure zero show that the test functions with support disjoint with $F$ are dense in $X$. (3) Use an analogous approach to that in Problem 26.*

## Problems to Section V.3 – spectrum of an unbounded operators

**Problem 28.** Consider the operator $M_{\boldsymbol{z}}$ from Problem 18.

(1) Show that $\sigma(M_{\boldsymbol{z}}) = \overline{\{z_n; n \in \mathbb{N}\}}$.
(2) Show that the eigenvalues of $M_{\boldsymbol{z}}$ are exactly the values $z_n$, $n \in \mathbb{N}$.
(3) Show that $\sigma(M_{\boldsymbol{z}})$ can be any nonempty closed subset of $\mathbb{C}$.

**Problem 29.** Consider the operator $M_g$ from Problem 20.

(1) Show that $\sigma(M_g)$ is the essential range of $g$.
(2) Show that $\lambda \in \mathbb{C}$ is an eigenvalue of $M_g$ if and only if $\mu(g^{-1}(\lambda)) > 0$.
(3) Show that, in case $(\Omega, \Sigma, \mu)$ is the interval $[0, 1]$ with the Lebesgue measure, $\sigma(M_g)$ can be any nonempty closed subset of $\mathbb{C}$.

**Problem 30.** Let $S, T$ be two closed operators on a Banach space $X$ such that $S \subsetneq T$.

(1) Show that any eigenvalue of $S$ is also an eigenvalue of $T$, but the converse need not hold.
(2) Show that $\rho(S) \cap \rho(T) = \emptyset$, in other words $\sigma(S) \cup \sigma(T) = \mathbb{C}$.

`Hint:` *(1) To find a counterexample look at Problem 31.*

**Problem 31.** Compute the eigenvalues and the spectrum of the following operators:

(1) The operator $T$ from Problem 22.
(2) The operator $T$ on $\ell^p(\mathbb{Z})$ (where $p \in [1, \infty)$) defined by

$$T((x_n)_{n\in\mathbb{Z}}) = (|n| \cdot x_n)_{n\in\mathbb{Z}}, \quad (x_n) \in D(T) = \{(y_n) \in \ell^p(\mathbb{Z}); (|n|\, y_n) \in \ell^p(\mathbb{Z}) \ \&\ \sum_{n\in\mathbb{Z}} y_n = 0\}.$$

(3) The operator $T$ from Problem 23.
(4) The operator $T$ from Problem 24.

**Problem 32.** Compute the eigenvalues and the spectrum of the operators from Problem 25.

`Hint:` *Use the standard methods of solving differential equations.*

## Problems to Section V.4 - operators on a Hilbert space

**Problem 33.** Let $T$ be a closed densely defined operator. Show that
$$\sigma(T^*) = \{\overline{\lambda}; \lambda \in \sigma(T)\}.$$

`Hint:` *Use Lemma V.25.*

**Problem 34.** Consider the operator $M_{\boldsymbol{z}}$ from Problem 18 on $\ell^2$. Show that $(M_{\boldsymbol{z}})^* = M_{\overline{\boldsymbol{z}}}$ and characterize sequences $\boldsymbol{z}$ for which $M_{\boldsymbol{z}}$ is selfadjoint.

**Problem 35.** Consider the operator $M_g$ from Problem 20 on $L^2(\mu)$. Show that $(M_g)^* = M_{\overline{g}}$ and characterize functions $g$ for which $M_g$ is selfadjoint.

**Problem 36.** Consider the operators $T_j$, $j = 1, \dots, 6$, from Problem 25 on $L^2((0,1))$.

(1) Show that $T_1^* = -T_4$, $T_2^* = -T_3$, $T_3^* = -T_2$, $T_4^* = -T_1$, $T_5^* = -T_5$, $T_6^* = -T_6$.
(2) Deduce that $iT_5$ and $iT_6$ are selfadjoint and that $iT_4$ is symmetric.

`Hint:` *(1) To prove the inclusions '$\supset$' use integration by parts. To prove '$\subset$' proceed as follows: Let $g \in D(T_j^*)$. Then there is $h \in L^2((0,1))$ such that $\langle T_j f, g\rangle = \langle f, h\rangle$ for any $f \in D(T_j)$. Set $H(t) = \int_0^t h$, $t \in [0,1]$. Apply integration by parts. Note that $\mathscr{D}((0,1)) \subset D(T_j)$ and deduce that the distributive derivative of $g + H$ on $(0,1)$ is zero, thus $g + H$ is almost everywhere equal to a constant. So, $g \in AC([0,1])$ and $H = g(0) - g$. Plug this to the computation and conclude.*

**Problem 37.** Let $T$ be a symmetric operator on a Hilbert space which is not maximal symmetric (i.e., it admits a proper symmetric extension). Show that $\sigma(T) = \mathbb{C}$.

`Hint:` *Combine the ideas from Problem 30 and Lemma V.29.*

**Problem 38.** Consider the operators $T_1$ and $T_2$, from Problem 26 on $L^2((0,\infty))$.

(1) Show that $T_1^* = -T_2$ and $T_2^* = -T_1$.
(2) Deduce that $iT_2$ is symmetric.
(3) Compute the eigenvalues and the spectrum of $T_1$ and $T_2$.
(4) Deduce that $iT_2$ is a maximal symmetric operator.

`Hint:` *(1) Proceed similarly as in Problem 36, use integration by parts on $[0,r]$, take the limit for $r \to \infty$ and use Problem 26(1). (3) To compute the eigenvalues use the standard methods of solving differential equations. To determine when $\lambda I - T_2$ is onto use first Lemma V.29 to show that, in case $\operatorname{Re}\lambda \neq 0$, the surjectivity is equivalent to the density of range. To determine when the range is dense use Proposition V.23 and the knowledge of eigenvalues of $T_1$. Finally, to describe $\sigma(T_1)$ use Problem 33. (4) Combine the result of (3) with Problem 37.*

**Problem 39.** Consider the operators $T_1, T_2$ and $T_3$, from Problem 27 on $L^2(\mathbb{R})$.

(1) Show that $T_1^* = -T_1$, deduce that $iT_1$ is selfadjoint and $T_2$, $T_3$ are symmetric.
(2) Show that
$$D(T_2^*) = \{f \in L^2(\mathbb{R}); \forall R > 0 : f \text{ is absolutely continuous both on } (0,R) \text{ and on } (-R,0) \;\&\; f' \in L^2(\mathbb{R})\}$$
and that $T_2^*(f) = -f'$ for $f \in D(T_2^*)$. (The derivative is taken in the sense "almost everywhere", not in the sense of distributions.)
(3) Show that
$$D(T_3^*) = \{f \in L^2(\mathbb{R}); f \text{ is absolutely continuous on each bounded open interval disjoint with F} \;\&\; f' \in L^2(\mathbb{R})\}$$

and that $T_3^*(f) = -f'$ for $f \in D(T_3^*)$. (The derivative is taken in the sense "almost everywhere", not in the sense of distributions.)
(4) Compute the eigenvalues and the spectrum of $T_1$.
(5) Determine the eigenvalues and the spectrum of $T_2$ and $T_3$.

`Hint:` *(1) Proceed similarly as in Problem 38. (2) Proceed similarly as in Problem 38 separately on $(0,\infty)$ and on $(-\infty,0)$. (3) $\mathbb{R}\setminus\mathbb{C}$ is an open set, so it is a countable disjoint union of open intervals. Proceed on each of these intervals separately - on bounded ones similarly as in Problem 36, on unbounded ones similarly as in Problem 38. (4) Use (1), Theorem V.30 and standard methods of solving differential equations. (5) Use (1), (4) and Problem 30.*

**Problem 40.** Consider the operator $T$ from Problem 22 on $\ell^2$.

(1) Define

$$\psi((x_n)) = \lim_{n\to\infty} nx_n, \qquad (x_n) \in D(\psi) = \{(y_n) \in \ell^2; \lim_{n\to\infty} ny_n \text{ exists and is finite}\}.$$

Show that $\psi$ is a densely defined linear functional on $\ell^2$.

(2) Show that

$$D(T^*) = \{(x_n) \in D(\psi); (x_k - \psi((x_n)))_{k=1}^\infty \in \ell_2\}$$

and that

$$T^*((x_n)) = (x_k - \psi((x_n)))_{k=1}^\infty, \quad (x_n) \in D(T^*).$$

`Hint:` *(2) The inclusion '$\supset$' can be proved by an easy computation starting from definitions. To show '$\subset$' proceed as follows: Let $\boldsymbol{y} \in D(T^*)$. Then there exists $\boldsymbol{z} \in \ell^2$ such that $\langle T\boldsymbol{x}, \boldsymbol{y}\rangle = \langle \boldsymbol{x}, \boldsymbol{z}\rangle$ for $\boldsymbol{x} \in D(T)$. Apply for $\boldsymbol{x} = \boldsymbol{e}_1 - \boldsymbol{e}_n$, $n \geq 2$, and deduce that $z_n = z_1 - y_1 + ny_n$, $n \geq 2$. Using the assumption that $\boldsymbol{z} \in \ell_2$ and hence $z_n \to 0$ compute $\lim ny_n$ and conclude.*

**Problem 41.** Compute $T^*$ for the operator $T$ from Problem 31(2) on $\ell^2(\mathbb{Z})$.

`Hint:` *Proceed similarly as in Problem 40.*

**Problem 42.** Consider the operator $T$ from Problem 23 on $L^2((1,\infty))$.

(1) Set

$$D(\psi) = \{f \in L^2((1,\infty)); \exists C \in \mathbb{C} : C + \varphi \cdot f \in L^2((1,\infty))\}.$$

Show that $D(\psi)$ is a dense linear subspace of $L^2((1,\infty))$, that for $f \in D(\psi)$ the constant $C$ from the definition is uniquely determined and the mapping

$$\psi : f \mapsto \text{ the respective } C$$

is a linear functional defined on $D(\psi)$.

(2) Show that $D(T^*) = D(\psi)$ and $T^*(f) = \psi(f) + \varphi \cdot f$ for $f \in D(T^*)$.

`Hint:` *The inclusion '$\supset$' can be proved by an easy computation starting from definitions. To show '$\subset$' proceed as follows: Let $g \in D(T^*)$. Then there exists $h \in L^2((1,\infty))$ such that $\langle Tf, g\rangle = \langle f, h\rangle$ for $f \in D(T)$. Apply for $f = (r-1)\chi_{(1,2)} - \chi_{(1,r)}$. Differentiate the resulting equality with respect to $r$ (use the known fact on differentiating the indefinite integral) and deduce that $h(r) = \int_1^2 (h - fg) + \varphi(r) \cdot g(r)$ almost everywhere. Then complete the argument.*

**Problem 43.** Compute $T^*$ for the operator $T$ from Problem 24 on $L^2((1,\infty))$.

`Hint:` *Proceed similarly as in Problem 42.*

**Problem 44.** Set $\varphi(t) = t$ for $t \in \mathbb{R}$. Compute the adjoint of the operator $T$ on $L^2(\mathbb{R})$ defined by

$$T(f) = \varphi \cdot f, \quad f \in D(T) = \{g \in L^2(\mathbb{R}); \varphi \cdot g \in L^2(\mathbb{R}) \ \& \int_{-\infty}^{\infty} f(t)e^{it} = 0\}.$$

`Hint:` *Proceed similarly as in Problem 42.*

## Problems to Section V.5 - symmetric operators and Cayley transform

**Problem 45.** Let $S$ be a self-adjoint operator on a Hilbert space and let $C_S$ be its Cayley transform.

(1) Show that $S$ is bounded (i.e., everywhere defined) if and only if $1 \notin \sigma(C_S)$.
(2) Suppose that $S$ is bounded. Describe $C_S$ using continuous functional calculus applied to $S$. And, conversely, describe $S$ using continuous functional calculus applied to $C_S$.
(3) Suppose $S$ is bounded. Using the descriptions in (2) describe the relationship of $\sigma(S)$ and $\sigma(C_S)$.
(4) Show that the relationship found in (3) is valid also in case $S$ is unbounded.

`Hint:` *(4) For $\lambda \neq 1$ show by a direct computation that $\lambda I - C_S$ is invertible if and only if $(\lambda - 1)S + i(\lambda + 1)I$ is one-to-one and surjective.*

**Problem 46.** Let $S$ be a closed densely defined symmetric operator on a Hilbert space. Show that there are the following posibilites for the spectrum of $S$.

(a) $\sigma(S) \subset \mathbb{R}$ if $S$ is selfadjoint.
(b) $\sigma(S) = \mathbb{C}$ if $S$ is not maximal.
(c) $\sigma(S) = \{\lambda \in \mathbb{C}; \operatorname{Im} \lambda \geq 0\}$ or $\sigma(S) = \{\lambda \in \mathbb{C}; \operatorname{Im} \lambda \leq 0\}$ if $S$ is maximal symmetric but not selfadjoint.

`Hint:` *To show (a) and (b) use Theorem V.30 and Problem 37. Assume that $S$ is maximal but not self-adjoint. Using Corollary V.31 and the remark (2) at the end of Section V.5 to show that $\sigma(S)$ contains exactly one of the numbers $\pm i$. Applying this observation to the operator $\frac{1}{\beta}(\alpha I - S)$ deduce that $\sigma(S)$ contains exactly one of the numbers $\alpha \pm \beta i$ whenever $\alpha \in \mathbb{R}$ and $\beta > 0$. Conclude using closedness of $\sigma(S)$ and connectedness of halfplanes.*

**Problem 47.** Consider the operators $M_g$ from Problem 20 on $L^2(\mu)$.

(1) Suppose $M_g$ is self-adjoint. Show that its Cayley transform is again of the form $M_h$ and determine $h$.
(2) Characterize functions $g$ for which $M_g$ is a unitary operator such that $I - M_g$ is one-to-one.
(3) Let $g$ be as in (2). Find $h$ such that $M_g$ is the Cayley transform of $M_h$.

**Problem 48.** Compute the Cayley transforms of the following operators:

(1) The operators $iT_5$ and $iT_6$, where $T_5$ and $T_6$ are the operators from Problem 25 on $L^2((0,1))$ (by Problem 36 they are selfadjoint).
(2) The operator $iT_4$, where $T_4$ is the operator from Problem 25 on $L^2((0,1))$ (by Problem 36 it is symmetric).
(3) The operator $iT_2$, where $T_2$ is the operator from Problem 26 on $L^2((0,\infty))$ (by Problem 38 it is symmetric).
(4) The operator $iT_1$, where $T_1$ is the operator from Problem 27 on $L^2(\mathbb{R})$ (by Problem 39 it is selfadjoint).
(5) The operator $iT_2$ and $iT_3$, where $T_2$ and $T_3$ are the operators from Problem 27 on $L^2(\mathbb{R})$ (by Problem 39 they are symmetric).

`Hint:` *The solution has two parts – to determine the domain of the Cayley transform and to compute the formula. The formula can be computed in all the cases using standard methods of solving differential equations and differentianting indefinite integrals. The domain in cases (1,4) is the whole space by Theorem V.35. In the remaing cases the domain can be described by* $D(C_S) = R(S + iI) = \operatorname{Ker}(S^* - iI)^\perp$ *(use Theorem V.32(a,c) and Proposition V.23). To describe it more concretely, use the knowledge of the spectrum and eigenvalues of the respective operators. The range of the Cayley transform can be determined similarly.*

**Problem 49.** Compute the Cayley transforms of the following operators:

(1) The operator $T$ from Problem 22 on $\ell^2$.
(2) The operator $T$ from Problem 23 on $L^2((1,\infty))$.
(3) The operator $T$ from Problem 24 on $L^2((1,\infty))$.

`Hint:` *The formula for the Cayley transform may de deduced using Problem 47. The domains can be found using the knowledge of spectra similarly as in Problem 48.*