## Lemma X.6

(a) Let $A$ be a Banach algebra with unit $e$. If $x \in A$ s.t. $\|x\| < 1$, then $e-x$ is invertible

and $$(e-x)^{-1} = \sum_{n=0}^{\infty} x^n \quad \left(= e + x + x^2 + x^3 + \dots\right),$$

where the series converges absolutely

Proof: $\|x^n\| \leq \|x\|^n$. (as $\|x \cdot y\| \leq \|x\| \cdot \|y\|$)

Since $\|x\| < 1$, $\sum_{n=0}^{\infty} \|x\|^n < \infty$, so $\sum_{n=0}^{\infty} \|x^n\| < \infty$.

Hence, the series converges absolutely. By completeness of $A$ it converges. Set $y := \sum_{n=0}^{\infty} x^n$

Then $$y \cdot x = \left(\lim_{N\to\infty} \left(\sum_{n=0}^{N} x^n\right)\right) \cdot x \underset{\uparrow}{=} \lim_{N\to\infty} \sum_{n=0}^{N} x^{n+1} =$$

as multiplication is cts (by Prop 4(b))

$$= \sum_{n=0}^{\infty} x^{n+1} = y - e$$

Similarly $x \cdot y = \sum_{n=0}^{\infty} x^{n+1} = y - e$

So, $$\left.\begin{aligned} y\cdot(e-x) &= y - yx = e \\ (e-x)\,y &= y - xy = e \end{aligned}\right\} \Rightarrow y = (e-x)^{-1}$$

(b) $x \in G(A)$, $h \in A$ s.t. $\|h\| < \frac{1}{\|x^{-1}\|}$

Then $x + h \in G(A)$, $(x+h)^{-1} = x^{-1} \cdot \sum_{n=0}^{\infty} (-1)^n (h \cdot x^{-1})^n$

and $$\|(x+h)^{-1} - x^{-1}\| \leq \frac{\|x^{-1}\|^2 \, \|h\|}{1 - \|x^{-1}\| \, \|h\|}$$

Proof: $\|h\| < \frac{1}{\|x^{-1}\|} \Rightarrow \|h \cdot x^{-1}\| \leq \|h\| \cdot \|x^{-1}\| < 1$

So, by (a) we get $e + hx^{-1} \in G(A)$ and

$$(e + hx^{-1})^{-1} = \sum_{n=0}^{\infty} (-1)^n (hx^{-1})^n$$

Further, $x + h = (e + hx^{-1}) \cdot x \in G(A)$
(by Prop. 5(a))

and $(x+h)^{-1} = x^{-1}(e + hx^{-1})^{-1} = x^{-1} \cdot \sum_{n=0}^{\infty} (-1)^n (hx^{-1})^n$

Moreover, $\|(x+h)^{-1} - x^{-1}\| = \|x^{-1} \cdot \sum_{n=1}^{\infty} (-1)^n (hx^{-1})^n\| \leq$

$\leq \|x^{-1}\| \cdot \|\sum_{n=1}^{\infty} (-1)^n (hx^{-1})^n\| \leq \|x^{-1}\| \cdot \sum_{n=1}^{\infty} \|(-1)^n (hx^{-1})^n\|$

$\leq \|x^{-1}\| \cdot \sum_{n=1}^{\infty} (\|h\| \cdot \|x^{-1}\|)^n = \|x^{-1}\| \cdot \frac{\|h\| \cdot \|x^{-1}\|}{1 - \|h\| \cdot \|x^{-1}\|}$