Counterexamples related to unbounded operators

① Densely defined operator without closed extension (Example XII. 7(5))

$X = \ell^2$, $D(T) = c_{00}$ (= sequences with only finitely many nonzero coordinates)
$T((x_n)) = \left(\sum_{n=1}^{\infty} x_n, 0, 0, 0, \dots\right)$

- Clearly $D(T)$ is a dense subspace of X
- Assume S is a closed extension of T

Let $x^k = (\underbrace{\tfrac{1}{k}, \dots, \tfrac{1}{k}}_{k \text{ times}}, 0, 0, 0, \dots)$. Then $x^k \in D(T) \subset D(S)$

$\|x^k\| = \sqrt{k \cdot \frac{1}{k^2}} = \frac{1}{\sqrt{k}} \xrightarrow{k\to\infty} 0$. So, $x^k \to 0$ in $D(S)$

Further, $Sx^k = Tx^k = (1, 0, 0, 0, \dots)$

S closed $\Rightarrow S0 = (1, 0, 0, 0, \dots)$, a contradiction.

② It may happen that $T(R+S) \supsetneqq TR + TS$ even if R, S are bounded and T is closed densely defined

$X = \ell^2$, $R = I$ (identity), $S((x_n)) = (-x_1, 0, -x_3, 0, -x_5, 0, \dots)$

Then $R, S \in L(X)$, $\|R\| = \|S\| = 1$

$D(T) = \{ (x_n) \in \ell^2 ;\ \sum_{n=1}^{\infty} n^2 |x_n|^2 < \infty \}$.

Then $D(T)$ is a dense subspace of ℓ^2 (it contains c_{00})

Define $T((x_n)) = (n x_n)$, $(x_n) \in D(T)$

- T is well defined, as $D(T) = \{ (x_n) \in \ell^2 ;\ (n x_n) \in \ell^2 \}$
- T is closed: assume $(x^k) \subset D(T)$, $x^k \to x$, $Tx^k \to y$ (in ℓ^2)

The convergence in ℓ^2 implies the coordinatewise convergence, hence $\forall n \in \mathbb{N}$: $x_n = \lim_{k\to\infty} x^k_n$

and $y_n = \lim_{k\to\infty} (Tx^k)_n = \lim_{k\to\infty} n x^k_n = n x_n$

Since $y\in\ell^2$, we deduce that $x\in D(T)$ and $y=Tx$

Now compare $D(T(R+S))$ and $D(TR+TS)$:

$$D(TR+TS)=D(TR)\cap D(TS)=\underbrace{\{x\in \underbrace{D(R)}_{\ell^2};\ \underbrace{Rx}_{=x}\in D(T)\}}_{=D(T)}\cap\{x\in\underbrace{D(S)}_{\ell^2};\ \underbrace{Sx}_{(-x_1,0,-x_3,0,-x_5,0,\dots)}\in D(T)\}$$

Observe: $x\in D(T)\Rightarrow(-x_1,0,-x_3,0,-x_5,0,\dots)\in D(T)$

So, $D(TR+TS)=D(T)$

But $$D(T(R+S))=\{x\in\underbrace{D(R+S)}_{=\ell^2};\ \underbrace{(R+S)x}_{=(0,x_2,0,x_4,0,x_6,0,\dots)}\in D(T)\}=\{x\in\ell^2;\ \sum_{n=1}^{\infty}(2n)^2x_{2n}^2<\infty\}\supsetneq D(T)$$

$(1,0,\frac{1}{3},0,\frac{1}{5},0,\dots)$ witnesses that.

(3) T densely defined, closed, $S\in L(X)$, ST has no closed extension (Example XII.12(1))

$X=C[0,1]$, $S(f)=\sum_{n=1}^{\infty}\frac{1}{2^n}f(\frac{1}{n})$ (the result is a constant function)

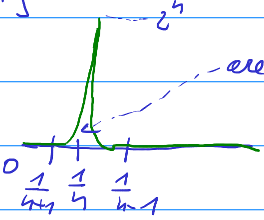
Then $S\in L(X)$, $\|S\|=1$

$D(T)=C^1[0,1]$, $Tf=f'$. Then T is densely defined (polynomials are dense) and closed (as $f_n\rightrightarrows f$, $f_n'\rightrightarrows g\Rightarrow f'=g$)

$D(ST)=D(T)=C^1[0,1]$

$ST(f)=\sum_{n=1}^{\infty}\frac{1}{2^n}f'(\frac{1}{n})$, $f\in C^1[0,1]$

For $n\in\mathbb{N}$ choose $g_n\in C[0,1]$ of the form:



area $\frac{1}{n}$, i.e. $\int_0^1 g_n=\frac{1}{n}$

Let $f_n(t)=\int_0^t g_n$, $t\in[0,1]$. Then $f_n\in C^1[0,1]$, $f_n'=g_n$

$$\left.\begin{aligned}&\|f_n\|=\tfrac{1}{n}\to 0,\text{ so } f_n\to 0\\ &STf_n=f_n'(\tfrac{1}{n})=g_n(\tfrac{1}{n})=1\end{aligned}\right\}\Rightarrow ST \text{ has no closed extension.}$$

(4) A variant on ℓ^2:

$X = \ell^2$, T as in (2), $S((x_n)) = \left(\sum_{n=1}^{\infty} \frac{x_n}{n}, 0, 0, 0, \dots\right)$

Then $S \in \mathcal{L}(X)$: $\|Sx\| = \left|\sum_{n=1}^{\infty} \frac{x_n}{n}\right| \le \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2} \cdot \|x\|$

$D(ST) = D(T)$, $STx = \left(\sum_{n=1}^{\infty} x_n, 0, 0, \dots\right)$, it has no closed extension by (1).

(5) S, T closed, densely defined, $S+T$ has no closed extension (Example XII.12(2))

$X = \ell^2$, $Y = \{(x_n) \in \ell^2 ; \sum_{n=1}^{\infty} n^2 |x_n|^2 < \infty\}$ $\quad$ ($= D(T)$ from (2))

Then Y is dense in X.

$T((x_n)) = (0, x_1, 2x_2, 3x_3, 4x_4, \dots)$, $(x_n) \in D(T) = Y$

Then T is closed (similarly as in (2))

$S((x_n)) = \left(\sum_{n=1}^{\infty} x_n, -x_1, -2x_2, -3x_3, -4x_4, \dots\right)$, $(x_n) \in D(S) = Y$

- S well defined : $\sum_{n=1}^{\infty} |x_n| = \sum_{n=1}^{\infty} \frac{1}{n} \cdot n|x_n| \le \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2} \cdot \left(\sum_{n=1}^{\infty} n^2 |x_n|^2\right)^{1/2} < \infty$ for $(x_n) \in Y$
- $(x_n) \in Y \Rightarrow$ the first coordinate is well defined and $S(x_n) \in \ell^2$
- S closed : $(x^k) \subset Y$, $x^k \to x$, $Sx^k \to y$ in ℓ^2

  as in (2) we deduce, using coordinatewise convergence that $y_n = -(n-1)x_{n-1}$ for $n \ge 2$. In particular, $x \in Y$ (as $y \in \ell^2$)

  Moreover,

$$\left|\sum_{n=1}^{\infty} x_n^k - \sum_{n=1}^{\infty} x_n\right| = \left|\sum_{n=1}^{\infty} \frac{1}{n} \cdot n(x_n^k - x_n)\right| \le$$

$$\le \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2} \cdot \underbrace{\left(\sum_{n=1}^{\infty} |n x_n^k - n x_n|^2\right)^{1/2}}_{\le \|Sx^k - y\| \to 0} \to 0$$

So, $y_1 = \lim_k \sum_{n=1}^{\infty} x_n^k = \sum_{n=1}^{\infty} x_n$.

Hence $y = Sx$

Finally, $(S+T)(x) = \left(\sum_{n=1}^{\infty} x_n, 0, 0, 0, \dots\right)$, $x \in Y$ has no closed extension by (1)