Prop. XII.18 T densely defined $\Rightarrow$ $\operatorname{Ker}(T^*) = R(T)^\perp$

Proof: $y \in \operatorname{Ker} T^* \Leftrightarrow y \in D(T^*)$ & $T^* y = 0$

$\Leftrightarrow$ $x \longmapsto \langle Tx, y \rangle$ is the zero functional on $D(T)$

$\Leftrightarrow$ $\forall x \in D(T): \langle Tx, y \rangle = 0 \Leftrightarrow y \in R(T)^\perp$

Lemma XII.19. $V : H \times H \to H \times H$, $V(x,y) = (-y, x)$

(a) V is a unitary operator on $H \times H$

clearly V is a linear bijection of $H \times H$ onto $H \times H$

$(V^{-1}(x,y) = (y, -x))$

clearly V is an isometry: $\| V(x,y) \| = \| (-y, x) \| = \sqrt{\|y\|^2 + \|x\|^2} = \| (x,y) \|$

So, V is unitary by Prop XI.17 (c)

(b) T densely defined on H $\Rightarrow$ $G(T^*) = (V(G(T)))^\perp = V (G(T)^\perp)$.

- the second equality follows from (a), unitary operators preserve orthogonality
- the first equality: $(u, v) \in (V(G(T)))^\perp \Leftrightarrow \forall (x,y) \in G(T): (u,v) \perp V(x,y)$

$\Leftrightarrow \forall (x,y) \in G(T): (u,v) \perp (-y, x) \Leftrightarrow \forall x \in D(T): (u,v) \perp (-Tx, x)$

$\Leftrightarrow \forall x \in D(T): \langle -Tx, u \rangle + \langle x, v \rangle = 0$

$\Leftrightarrow \forall x \in D(T): \langle x, v \rangle = \langle Tx, u \rangle \Leftrightarrow u \in D(T^*)$ & $v = T^* u$

$\Leftarrow$ by definition of T^*

$\Rightarrow$ $x \mapsto \langle x, v \rangle$ is cts $\Rightarrow u \in D(T^*)$

$\langle Tx, u \rangle$

by definition of T^* we get $T^* u = v$

$\Leftrightarrow (u, v) \in G(T^*)$

Lemma XII.20 T densely defined, one-to-one, $R(T)$ dense

$\Rightarrow$ T^* is one-to-one and $(T^{-1})^* = (T^*)^{-1}$

Proof: Assume T is densely defined, one-to-one, $R(T)$ dense.

$\Rightarrow T^{-1}$ is well-defined, $D(T^{-1}) = R(T)$ is dense

So, both T^* and $(T^{-1})^*$ are defined.

Moreover, $\ker T^* \overset{P18}{=} R(T)^\perp = \{0\}$ (as $R(T)$) is dense

$\Rightarrow$ T^* is one-to-one, $(T^*)^{-1}$ exists

Let V be the unitary operator from L XII.19.

Define $U(x,y) = (y,x)$. Then U is also a unitary operator on $H \times H$,

moreover $G(T^{-1}) = U(G(T))$

Observe: $UV = -VU$ $\quad\left[\begin{array}{l} UV(x,y) = U(-y,x) = (x,-y) \\ VU(x,y) = V(y,x) = (-x,y) \end{array}\right]$

So: $G((T^*)^{-1}) = U(G(T^*)) = U(V(G(T)^\perp)) \overset{U,V \text{ unitary}}{=} (UV(G(T)))^\perp$

$\overset{UV=-VU}{=} (-VU(G(T)))^\perp \overset{A^\perp = (-A)^\perp}{=} (VU(G(T)))^\perp = (V(G(T^{-1})))^\perp = G((T^{-1})^*)$

Prop. XII.21 T densely defined

(a) T^* is closed

$\Big[$ L19 $\Rightarrow$ $G(T^*) = (\text{something})^\perp$, so it is closed $\Big]$

(b) T has a closed extension $\Leftrightarrow$ T^* is densely defined

(then $\overline{T} = T^{**}$)

$\ulcorner \Leftarrow$: T^* densely defined $\Rightarrow$ T^{**} is defined. By L1d we have

$$G(T^{**}) = \left(V(G(T^*))\right)^\perp = \left(V\left(V(G(T))^\perp\right)\right)^\perp =$$

$$= \left(V(V(G(T)))\right)^{\perp\perp} = \left(-G(T)\right)^{\perp\perp} = G(T)^{\perp\perp} = \overline{G(T)}$$

V unitary

$V^2 = -I$

$V(V(x,y)) = V(-y,x) = (-x,-y)$

$(-A)^\perp = A^\perp$

$A^{\perp\perp} = \overline{\text{span} A}$

So, $T^{**} = \overline{T}$ is a closed extension of T

$\Rightarrow$: T^* not densely defined $\Rightarrow$ $\exists\, y \in H \setminus \{0\}$ $y \in D(T^*)^\perp$

Then

$$(y,0) \in G(T^*)^\perp = V\left(G(T)^\perp\right)^\perp = V\left(G(T)^{\perp\perp}\right)$$

$G(T)^{\perp\perp} \subset H \oplus H$

$$\Rightarrow (0,y) = V(y,0) \in V\left(V\left(G(T)^{\perp\perp}\right)\right) = -G(T)^{\perp\perp} = G(T)^{\perp\perp}$$

$V^2 = -I$

$A^{\perp\perp} = \overline{\text{span} A}$, $\parallel$ $\overline{G(T)}$

Hence $(0,y) \in \overline{G(T)}$, $y \neq 0$ $\Rightarrow$ T has no closed extension (P XII.10 (3)) $\lrcorner$

(c) T closed $\Leftrightarrow$ $T = T^{**}$

$\ulcorner \Leftarrow$: T^{**} is closed by (a)

$\Rightarrow$ T closed $\Rightarrow$ T has a closed extension (namely T) $\overset{(b)}{\Rightarrow}$ T^* densely defined & $T = \overline{T} = T^{**}$ $\lrcorner$