Spectral measure, measurable calculus:

① Let H be a Hilbert space and $T \in \mathcal{L}(H)$ a normal operator. Let $f \mapsto \tilde{f}(T)$, $f \in \mathcal{C}(\sigma(T))$, be the cts functional calculus.

Fix $x, y \in H$. Then $f \mapsto \langle \tilde{f}(T)x, y \rangle$ is a linear functional on $\mathcal{C}(\sigma(T))$. Moreover,

$$|\langle \tilde{f}(T)x, y\rangle| \leq \|\tilde{f}(T)\| \, \|x\| \, \|y\| \leq \|f\|_{\infty} \|x\| \, \|y\|$$

So, the norm of this functional is $\leq \|x\| \cdot \|y\|$

Hence, by the Riesz representation theorem $\exists!\, E_{x,y}$, a complex Borel measure on $\sigma(T)$ s.t.

$$\langle \tilde{f}(T)x, y\rangle = \int_{\sigma(T)} f \, dE_{x,y}, \quad f \in \mathcal{C}(\sigma(T))$$

Moreover, $\|E_{x,y}\| \leq \|x\| \cdot \|y\|$. (This proves Prop. 2 (d))

② $x \mapsto E_{x,y}$ is linear for $y \in H$, $y \mapsto E_{x,y}$ is conjugate linear for $x \in H$

$f \in C(\sigma(T))$

$$\int_{\sigma(T)} f \, dE_{x_1+x_2,y} = \langle \tilde{f}(T)(x_1+x_2), y\rangle = \langle \tilde{f}(T)x_1, y\rangle + \langle \tilde{f}(T)x_2, y\rangle =$$

$$= \int_{\sigma(T)} f \, dE_{x_1,y} + \int_{\sigma(T)} f \, dE_{x_2,y} = \int_{\sigma(T)} f \, d(E_{x_1,y} + E_{x_2,y})$$

$$\int_{\sigma(T)} f \, dE_{\alpha x,y} = \langle \tilde{f}(T)(\alpha x), y\rangle = \alpha \langle \tilde{f}(T)x, y\rangle = \alpha \int_{\sigma(T)} f \, dE_{x,y} = \int_{\sigma(T)} f \, d(\alpha E_{x,y})$$

So, $x \mapsto E_{x,y}$ is linear. The case $y \mapsto E_{x,y}$ is similar

[This proves Prop. 2 (a, b)]

(3) $x \in H \Rightarrow E_{x,x}$ is a nonnegative measure

⌈ To prove this, it is enough to show

$$f \in \mathcal{C}(\sigma(T)),\ f \geq 0 \Rightarrow \int_{\sigma(T)} f\, dE_{x,x} \geq 0 \quad \text{(by the Riesz theorem)}$$

So, fix $f \in \mathcal{C}(\sigma(T))$, $f \geq 0$. Then $\sigma(\hat{f}(T)) = f(\sigma(T)) \subset [0, \infty)$. Moreover, since f is real-valued, i.e. $\bar{f} = f$, we get that $\hat{f}(T)$ is self-adjoint. Thus $\hat{f}(T)$ is a positive operator, so $\langle \hat{f}(T)x, x\rangle \geq 0$ for $x \in H$ (Prop. XII.5(c)) ⌋

So, Prop. 2 (c) is proved.

(4) $$E_{x,y} = \frac{1}{4}\left(E_{x+y,x+y} - E_{x-y,x-y} + i E_{x+iy,x+iy} - i E_{x-iy,x-iy}\right),$$

$x, y \in H$

(This is Prop. 2 (e))

⌈ This can be proved by a direct computation using just (a,b) (i.e. (2) above).

See also the proof of Lemma XII.2 ⌋

• So, Prop. 2 is proved.

⑤ Let $\mathcal{A}$ denote the σ-algebra of all the subsets of $\sigma(T)$ which are $E_{x,y}$ - measurable for each $x,y \in H$

[recall that A is $E_{x,y}$-measurable if there are Borel sets B, C s.t. $B \subset A \subset C$ and $|E_{x,y}|(C \setminus B) = 0$]

Note that $A \in \mathcal{A} \iff \forall x \in H$ A is $E_{x,x}$ -measurable

($\Rightarrow$ obvious $\Leftarrow$ by Prop. 2 (e), see ④ above)

⑥ Let $f: \sigma(T) \to \mathbb{C}$ be a bdd $\mathcal{A}$-measurable function.

Then $B_f(x,y) = \int_{\sigma(T)} f \, dE_{x,y}$ satisfies the assumptions of

Prop. 1 (the first two properties follow from Prop. 2 (a,b)), and for $x,y \in B_H$

$$|B_f(x,y)| = \left| \int_{\sigma(T)} f \, dE_{x,y} \right| \leq \int_{\sigma(T)} |f| \, d|E_{x,y}| \leq$$

$$\leq \|f\|_\infty \|x\| \cdot \|y\|$$, so $\|B_f\| \leq \|f\|_\infty$)

Thus by Prop. 1 $\exists! \, \tilde{f}(T) \in L(H)$ s.t.

$$\langle \tilde{f}(T)x, y \rangle = \int_{\sigma(T)} f \, dE_{x,y}, \quad x,y \in H$$

If f is cts, then this $\tilde{f}(T)$ coincides with the result of the cts functional calculus (by uniqueness)

The assignement $f \mapsto \tilde{f}(T)$ is called the <u>measurable calculus for T</u>.

⑦ $A \in \mathcal{A} \Rightarrow E_T(A) := \widetilde{\chi_A}(T)$

E_T is the <u>spectral measure of T</u>

(8) $\mathcal{N} = \{ A \in \mathcal{A} ;\ \forall x,y \in H\ |E_{x,y}|(A) = 0 \}$

Then $\mathcal{N} = \{ A \in \mathcal{A} ;\ \forall x \in H : E_{x,x}(A) = 0 \}$

($\subset$ obvious $\supset$ by Prop. 12 (d))

(9) f, g bdd $\mathcal{A}$-measurable, $\{ \lambda \in \sigma(T) ;\ f(\lambda) \neq g(\lambda) \} \in \mathcal{N}$
$\Rightarrow \tilde{f}(T) = \tilde{g}(T)$

$\ulcorner$ $x, y \in H \Rightarrow$ $f = g$ $|E_{x,y}|$-a.e.,

so $\langle \tilde{f}(T)x, y \rangle = \int_{\sigma(T)} f \, dE_{x,y} = \int_{\sigma(T)} g \, dE_{x,y} = \langle \tilde{g}(T)x, y \rangle$ $\lrcorner$

(10) Let $L^\infty(E_T)$ denote the space of equivalence classes of bdd $\mathcal{A}$-measurable functions, where f, g are equivalent iff $\{ \lambda ;\ f(\lambda) \neq g(\lambda) \} \in \mathcal{N}$

$[f] \in L^\infty(E_T)$ ... define $\| [f] \| := \operatorname{ess\,sup}_{\sigma(T)} |f| =$

$= \inf \{ c > 0 ;\ \{ \lambda \in \sigma(T) ;\ |f(\lambda)| > c \} \in \mathcal{N} \}$

Then $L^\infty(E_T)$ is a unital commutative C^*-algebra with the natural operations

(11) By (9) we see that the measurable calculus

$f \mapsto \tilde{f}(T)$ can be interpreted as a mapping $L^\infty(E_T) \to L(H)$

(12) It follows from Luzin theorem, that:

Let K be a compact metric space, μ a finite Borel measure on K, and $f: K \to \mathbb{C}$ be a bdd μ-measurable function (i.e. f is measurable w.r.t. to the completion of μ)

Then there is a sequence (f_n) of cts functions on K s.t.

- (f_n) is uniformly bdd
- $f_n \to f$ μ-a.e.

[$\exists L_n \subset K$ compact, $\mu(K \setminus L_n) < 2^{-n}$ & $f|_{L_n}$ cts. Let $f_n: K \to \mathbb{C}$ be a cts extension of $f|_{L_n}$ provided by Tietze theorem]

In partic., there is $g: K \to \mathbb{C}$ bdd Borel function s.t. $f = g$ μ-a.e.

[$g(t) = \begin{cases} \lim f_n(t) & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}$]

(13) H separable $\Rightarrow$ $\forall f$ bdd $\mathcal{A}$-measurable

- $\exists (f_n)$ a uniformly bdd sequence of cts functions s.t. $f_n \to f$ except for a set from $\mathcal{N}$
- $\exists g: \sigma(T) \to \mathbb{C}$ bdd Borel s.t. $f = g$ except on a set from $\mathcal{N}$

[$(x_n)_{n \in \mathbb{N}}$ dense in H, not containing 0

Then $\mathcal{N} = \{A \in \mathcal{A};\ E_{x_n, x_n}(A) = 0 \text{ for } n \in \mathbb{N}\}$

[$\subset$ obvious; $\supset$: $(x, y) \mapsto E_{x,y}(A)$ is cts by Prop. 2(d), so $x \mapsto E_{x,x}(A)$ is also cts]

Let $\mu := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{E_{x_n, x_n}}{\|x_n\|^2}$ $\Rightarrow$ μ is a finite Borel measure on $\sigma(T)$

and $\mathcal{N} = \mu$-null sets, so we can apply (12) to μ
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