PROPERTIES OF THE MEASURABLE CALCULUS (Thm XIII.4)

(14) $f \longmapsto \tilde{f}(T)$ is a linear mapping $\mathcal{L}^\infty(E_T) \to \mathcal{L}(H)$

⌈ obvious ⌋

(15) $\tilde{\bar{f}}(T) = \tilde{f}(T)^*$, $f \in \mathcal{L}^\infty(E_T)$

⌈ Let $x \in H$ be arbitrary. Then

$$\langle \tilde{\bar{f}}(T)x, x\rangle = \int_{\sigma(T)} \bar{f}\, dE_{x,x}$$

$$\langle \tilde{f}(T)^* x, x\rangle = \langle x, \tilde{f}(T)x\rangle = \overline{\langle \tilde{f}(T)x, x\rangle} =$$

$$= \overline{\int_{\sigma(T)} f\, dE_{x,x}} \underset{E_{x,x} \geq 0}{=} \int_{\sigma(T)} \bar{f}\, dE_{x,x} = \langle \tilde{\bar{f}}(T)x, x\rangle$$

So, $\tilde{f}(T)^* = \tilde{\bar{f}}(T)$ by Proposition XII.3(c) ⌋

(16) $\widetilde{fg}(T) = \tilde{f}(T)\,\tilde{g}(T)$, $f, g \in \mathcal{L}^\infty(E_T)$

⌈ • we know it works if f, g are cts

• Suppose g is cts.

Let $x, y \in H$

Let (f_n) be a uniformly bdd sequence of cts functions

s.t. $f_n \to f$ $\;|E_{\tilde{g}(T)x,y}| + |E_{x,y}|$ - a.e. (cf. L3(a))

Then $\langle \tilde{f}(T)\tilde{g}(T)x, y\rangle = \int_{\sigma(T)} f\, dE_{\tilde{g}(T)x,y} =$

$$\overset{(L)}{=} \lim_{n\to\infty} \int_{\sigma(T)} f_n\, dE_{\tilde{g}(T)x,y} = \lim_{n\to\infty} \langle \tilde{f_n}(T)\tilde{g}(T)x, y\rangle =$$

the valid cts for cts functions

$$= \lim_{n\to\infty} \langle \widetilde{f_n g}(T)x, y\rangle = \lim_{n\to\infty} \int f_n g\, dE_{x,y} \overset{(L)}{=} \int fg\, dE_{x,y} = \langle \widetilde{fg}(T)x, y\rangle$$

(L) = Lebesgue domin. conv. thm

- $f, g \in \mathcal{L}^\infty(E_T)$ general

  Let $x, y \in H$

  Let (g_n) be a unif. bdd sequence of cts functions s.t.

$$g_n \to g \quad |E_{x,y}| + |E_{x,\tilde{f}(T)^* y}| - a.e. \qquad (\text{cf. L3(a)})$$

$$\langle \tilde{f}(T)\tilde{g}(T)x, y\rangle = \langle \tilde{g}(T)x, \tilde{f}(T)^* y\rangle = \int_{\sigma(T)} g \, dE_{x, \tilde{f}(T)^* y} =$$

Lebesgue dom. conv. thm

$$= \lim_{n\to\infty} \int_{\sigma(T)} g_n \, dE_{x, \tilde{f}(T)^* y} = \lim_{n\to\infty} \langle \tilde{g_n}(T)x, \tilde{f}(T)^* y\rangle =$$

$$= \lim_{n\to\infty} \langle \tilde{f}(T)\tilde{g_n}(T)x, y\rangle = \lim_{n\to\infty} \langle \widetilde{fg_n}(T)x, y\rangle =$$

the previous case

$$= \lim_{n\to\infty} \int_{\sigma(T)} fg_n \, dE_{x,y} = \int_{\sigma(T)} fg \, dE_{x,y} = \langle \widetilde{fg}(T)x, y\rangle$$

Lebesgue domm. conv. thm. $\lrcorner$

(17) Summarizing (14), (15), (16): $f \mapsto \tilde{f}(T)$ is a $*$-homomorphism

$$\mathcal{L}^\infty(E_T) \to \mathcal{L}(H)$$

In part: f real-valued (except on a set from $\mathcal{N}$) $\Rightarrow \tilde{f}(T)$ self-adjoint

(18) $f \geq 0 \Rightarrow \tilde{f}(T) \geq 0$, Moreover $\tilde{f}(T) = 0 \Leftrightarrow f = 0$)

- $f \geq 0 \Rightarrow \langle \tilde{f}(T)x, x\rangle = \int f \, dE_{x,x} \geq 0$ as $E_{x,x} \geq 0$

  $\tilde{f}(T) = 0 \Rightarrow \forall x \ \langle \tilde{f}(T)x, x\rangle = 0 \Rightarrow \forall x \int f \, dE_{x,x} = 0$

  As $f \geq 0$, we deduce $f = 0$ $E_{x,x}$ - a.e.

  Hence $f = 0$ except on a set from $\mathcal{N}$

(19) $\tilde{f}(T) = 0 \Rightarrow f = 0$ except on a set from $\mathcal{N}$

Γ $f \geq 0 \Rightarrow$ by (18)

f real-valued $\Rightarrow f = f^+ - f^-$, $f^+, f^- \in L^\infty(E_T)$

Then $\tilde{f}(T) = \tilde{f^+}(T) - \tilde{f^-}(T)$, thus $\tilde{f^+}(T) = \tilde{f^-}(T)$.

So, $\widetilde{(f^+)^2}T \overset{(16)}{=} \tilde{f^+}(T)\tilde{f^+}(T) = \tilde{f^+}(T)\tilde{f^-}(T) =$

$\overset{(16)}{=} \widetilde{f^+f^-}(T) = \tilde{0}(T) = 0$

Thus $f^+ = 0$ except a set from $\mathcal{N}$ and the same holds for f^-, thus $f = 0$ except a set from $\mathcal{N}$

f complex $\Rightarrow \tilde{f}(T) = \underbrace{\widetilde{\operatorname{Re} f}(T)}_{\text{self-adjoint}} + i\,\widetilde{\operatorname{Im} f}(T)$

$\Rightarrow \widetilde{\operatorname{Re} f}(T) = 0$ & $\widetilde{\operatorname{Im} f}(T) = 0$

$\Rightarrow \operatorname{Re} f = 0$, $\operatorname{Im} f = 0$ except on a set from $\mathcal{N}$

$\Rightarrow f = 0$ except on a set from $\mathcal{N}$. $\lrcorner$

(20) So, $f \mapsto \tilde{f}(T)$ is a $*$-isomorphism, so it is an isometry.

So, Thm 4(a) is proved

<u>In particular:</u>
- $\tilde{f}(T)$ is always a normal operator
- $\tilde{f}(T)$ is self-adjoint $\Leftrightarrow f$ real valued (except a set from $\mathcal{N}$)
- $\sigma(\tilde{f}(T)) = \sigma(\tilde{f}) = \operatorname{ess\,rng}(f)$

$= \{\lambda \in \mathbb{C};\ \forall r > 0\ f^{-1}(U(\lambda, r)) \notin \mathcal{N}\}$

So, assertions (d) and (c) follow

(21) $(f_n) \subset \mathcal{L}^\infty(E_T)$, $f_n \to f$ pointwise except on a set from $\mathcal{N}$

(f_n) unif. bdd

$$\Rightarrow f \in \mathcal{L}^\infty(E_T), \qquad \langle \tilde{f_n}(T)x, y\rangle \to \langle \tilde{f}(T)x, y\rangle$$

[use definitions and Lebesgue dominated conv. thm] $\Rightarrow$ (b) holds

(22) $f \in \mathcal{L}^\infty(E_T)$, $g \in \mathcal{C}(\sigma(\tilde{f}(T))) = \mathcal{C}(\text{ess rng } f)$

$$\Rightarrow \widetilde{g \circ f}(T) = \tilde{g}(\tilde{f}(T))$$

$\lceil$ $A := \{ g \in \mathcal{C}(\sigma(\tilde{f}(T))) ; \ \widetilde{g \circ f}(T) = \tilde{g}(\tilde{f}(T)) \}$

- A is a linear subspace
- $1 \in A$, $id \in A$
- $g \in A \Rightarrow \bar{g} \in A$ (as $\bar{g} \circ f = \overline{g \circ f}$)
- $g_1, g_2 \in A \Rightarrow g_1 g_2 \in A$

  $\lceil$ $\widetilde{(g_1 g_2) \circ f}(T) = \widetilde{(g_1 \circ f) \cdot (g_2 \circ f)}(T) = \widetilde{g_1 \circ f}(T)\, \widetilde{g_2 \circ f}(T)$

  $= \tilde{g_1}(\tilde{f}(T))\, \tilde{g_2}(\tilde{f}(T)) = \widetilde{g_1 g_2}(\tilde{f}(T))$ $\rfloor\rfloor$

- A is norm-closed

So, by Stone-Weierstrass thm we get $A = \mathcal{C}(\sigma(\tilde{f}(T)))$ $\rfloor$

$\Rightarrow$ (e) is proved

(23) $ST = TS \Rightarrow S\tilde{f}(T) = \tilde{f}(T)S$

$\lceil$ • we know it works if f is cts

• f general: Fix $x, y \in H$. Find (f_n) unif. bdd sequence of cts functions

s.t. $f_n \to f$ $|E_{Sx,y}| + |E_{x,S^*y}|$ – a.e. (cf L3(a))

Then $\langle S\tilde{f}(T)x, y\rangle = \langle \tilde{f}(T)x, S^*y\rangle = \int f\, dE_{x,S^*y} = \lim_n \int f_n\, dE_{x,S^*y} =$

$= \lim_n \langle \tilde{f_n}(T)x, S^*y\rangle = \lim_n \langle S\tilde{f_n}(T)x, y\rangle = \lim_n \langle \tilde{f_n}(T)Sx, y\rangle =$

$= \lim_n \int f_n\, dE_{Sx,y} = \int f\, dE_{Sx,y} = \langle \tilde{f}(T)Sx, y\rangle$ $\rfloor$

So, (f) is proved