

Invertibility and spectrum in A and A^+

(1) $a \in A \Rightarrow (a, 0)$ is not invertible in A^+

$$\Gamma (a, 0) \cdot (b, \lambda) = (ab + \lambda a, 0) \neq (0, 1) \quad \square$$

(2) A has no unit $\Rightarrow \forall x \in A \quad \sigma(x) \ni 0$

$$\Gamma \text{ By definition } \sigma(x) = \sigma_{A^+}(x, 0) \ni 0, \text{ as } (x, 0) \text{ is not invertible by (1) } \square$$

(3) A has a unit $e \Rightarrow (a, \lambda) \in A^+$ is invertible iff $\lambda \neq 0$ & $a + \lambda e$ is invertible in A

$$\Gamma \text{ Recall } (a, \lambda) (b, \mu) = (ab + \lambda b + \mu a, \lambda \mu)$$

\Rightarrow Suppose that (a, λ) is invertible in A^+

$$\bullet \exists (b, \mu) \text{ s.t. } (a, \lambda) \cdot (b, \mu) = (0, 1)$$

in particular $\lambda \mu = 1$, hence $\lambda \neq 0$

Moreover,

$$\bullet \exists (b, \mu) \text{ s.t. } (a, \lambda) \cdot (b, \mu) = (e, 0)$$

$$\text{Hence } ab + \lambda b + \mu a = e$$

$$\lambda \mu = 0$$

Since $\lambda \neq 0$, necessarily $\mu = 0$.

$$\text{Thus } ab + \lambda b = e$$

$$(a + \lambda e)b$$

$\Rightarrow a + \lambda e$ has a right inverse in A

$$\bullet \text{ Similarly } \dots \exists (b, \mu) \text{ s.t. } (b, \mu) (a, \lambda) = (e, 0)$$

$$\text{Then } b(a + \lambda e) = e$$

$$b(a + \lambda e) \Rightarrow a + \lambda e \text{ has a left inverse in } A$$

So, $a + \lambda e$ is invertible in A

\Leftarrow : $\lambda \neq 0$, $a + \lambda e$ invertible in A . Let us look for (b, μ) with

$$(a, \lambda)(b, \mu) = (0, 1)$$

$$\text{then } \lambda\mu = 1 \Rightarrow \mu = \frac{1}{\lambda}$$

$$\text{and } 0 = ab + \lambda b + \mu a = ab + \lambda b + \frac{1}{\lambda}a$$

$$0 = (a + \lambda e)b + \frac{1}{\lambda}a \Rightarrow b = -\frac{1}{\lambda}(a + \lambda e)^{-1}a$$

Then, indeed, $(\frac{1}{\lambda}(a + \lambda e)^{-1}a, \frac{1}{\lambda})$ is a right inverse

Similarly, $(\frac{1}{\lambda}a \cdot (a + \lambda e)^{-1}, \frac{1}{\lambda})$ is a left inverse

SO (a, λ) is invertible in A^+ \lrcorner

$$\textcircled{1} \quad A \text{ has a unit } e \Rightarrow \sigma_{A^+}(a, 0) = \sigma_A(a) \cup \{0\}$$

More generally:

$$\lambda \in \sigma_{A^+}(a, \mu) \Leftrightarrow (-a, \lambda - \mu) \text{ is not invertible}$$

$$\textcircled{2} \quad \Leftrightarrow \lambda - \mu = 0 \quad \text{or} \quad (\lambda - \mu)e - a \text{ is not invertible}$$

$$\Leftrightarrow \lambda = \mu \quad \text{or} \quad \lambda - \mu \in \sigma_A(a)$$

$$\text{So, } \sigma_{A^+}(a, \mu) = \{\mu\} \cup (\mu + \sigma_A(a))$$

$$\text{Concretely, } \sigma_{A^+}(a, 0) = \{0\} \cup \sigma_A(a). \quad \lrcorner$$