Proposition X.15 Let A be a Banach algebra with unit e, $B \subset A$ a closed subalgebra s.t. $e \in B$. Let $x \in B$. Then:

(a) $\partial \sigma_B(x) \subset \sigma_A(x) \subset \sigma_B(x)$

(b) Let G be a connected component of $\mathbb{C} \setminus \sigma_A(x)$. Then either $G \subset \sigma_B(x)$ or $G \cap \sigma_B(x) = \emptyset$

(c) If $\mathbb{C} \setminus \sigma_A(x)$ is connected, then $\sigma_A(x) = \sigma_B(x)$

Proof: (1) $\sigma_A(x) \subset \sigma_B(x)$:

Let $\lambda \in \mathbb{C} \setminus \sigma_B(x) = \rho_B(x) \Rightarrow \lambda e - x$ is invertible in B, hence it is also invertible in A (the same inverse), so $\lambda \in \mathbb{C} \setminus \sigma_A(x)$.

(2) Let G be a connected component of $\mathbb{C} \setminus \sigma_A(x)$

Suppose that $\lambda_0 \in G \cap \sigma_B(x)$. Then $R(\lambda_0, x) \notin B$, so, by H-B theorem, $\exists \varphi \in A^*$ s.t. $\varphi(R(\lambda_0, x)) = 1$ & $\varphi|_B \equiv 0$ Theorem II.9

Then $\lambda \mapsto \varphi(R(\lambda, x))$ is holomorphic on G (Prop. 8(v)) and $\varphi(R(\lambda, x)) = 0$ for $\lambda \in G \setminus \sigma_B(x)$
[then $R(\lambda, x) \in B$]

$G \setminus \sigma_B(x)$ is an open set. If $G \setminus \sigma_B(x) \neq \emptyset$, it has accumulation points, so, by the uniqueness theorem $\varphi(R(\lambda, x)) = 0$ on G. It is a contradiction, as $\varphi(R(\lambda_0, x)) = 1$.

Thus $G \setminus \sigma_B(x) = \emptyset$, so $G \subset \sigma_B(x)$.

This completes the proof of (b).

(3) $\partial\sigma_B(x) \subset \sigma_A(x)$

$\ulcorner$ Let $\lambda \in \sigma_B(x) \setminus \sigma_A(x)$
Let $G \subset \mathbb{C} \setminus \sigma_A(x)$ be the component containing λ. By (3) we get $G \subset \sigma_B(x)$.
As G is open, $\lambda \in \operatorname{Int} \sigma_B(x)$, thus $\lambda \notin \partial\sigma_B(x)$ $\lrcorner$

(4) If $\mathbb{C} \setminus \sigma_A(x)$ is connected, then $\sigma_A(x) = \sigma_B(x)$

$\ulcorner$ $G := \mathbb{C} \setminus \sigma_A(x)$ is the only component. By (3) either $G \subset \sigma_B(x)$ or $G \cap \sigma_B(x) = \phi$.
The case $G \subset \sigma_B(x)$ is impossible, as it would imply $\sigma_B(x) = \mathbb{C}$. $\lrcorner$
as $\sigma_B(x)$ is bdd by Thm 9

<u>Corollary</u> X.16: A Banach-algebra, $B \subset A$ closed subalgebra, $x \in B$. Then (a)–(c) hold if we replace $\sigma_A(x)$, $\sigma_B(x)$ by $\sigma_A(x) \cup \{0\}$ and $\sigma_B(x) \cup \{0\}$

<u>Proof</u> Consider A^+ and define $\tilde{B} = \operatorname{span}(\{(b,0), b \in B\} \cup \{(0,1)\})$
Then $\tilde{B}$ is isomorphic to B^+

Hence $\sigma_{A^+}(x,0) = \sigma_A(x) \cup \{0\}$

$\sigma_{\tilde{B}}(x,0) = \sigma_B(x) \cup \{0\}$

and we can apply the previous proposition

to $(x,0) \in \tilde{B} \subset A^+$.