Holomorphic functional calculus

A ... a Banach algebra with unit e

$x \in A$, $\Omega \subset \mathbb{C}$ open set, $\Omega \supset \sigma(x)$

Γ ... a cycle around $\sigma(x)$ in Ω, i.e.

- Γ is a cycle in $\Omega \setminus \sigma(x)$
- $\operatorname{ind}_\Gamma z \in \{0,1\}$ for $z \in \mathbb{C} \setminus \langle\Gamma\rangle$
- $\operatorname{ind}_\Gamma z = 1$ for $z \in \sigma(x)$
- $\operatorname{ind}_\Gamma z = 0$ for $z \in \mathbb{C} \setminus \Omega$

The existence of Γ follows from complex analysis

Let f be a holomorphic function on Ω. Define

$$\tilde{f}(x) := \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda e - x)^{-1}\, d\lambda$$

① The integral exists in Bochner sense by Proposition 17, as $\lambda \mapsto f(\lambda)(\lambda e - x)^{-1}$ is cts on $\langle\Gamma\rangle$ by Prop. 8 (ii)

② The value of $\tilde{f}(x)$ does not depend on Γ:

Let Γ_1, Γ_2 be two cycles with the above properties.

Consider the cycle $\Gamma_1 \dot{-} \Gamma_2$. Then $\langle \Gamma_1 \dot{-} \Gamma_2 \rangle \subset \Omega \setminus \sigma(x)$

$\forall z \in \sigma(x)$ $\operatorname{ind}_{\Gamma_1 \dot{-} \Gamma_2} z = \operatorname{ind}_{\Gamma_1} z - \operatorname{ind}_{\Gamma_2} z = 1 - 1 = 0$

$\forall z \in \mathbb{C} \setminus \Omega$ $\operatorname{ind}_{\Gamma_1 \dot{-} \Gamma_2} z = \operatorname{ind}_{\Gamma_1} z - \operatorname{ind}_{\Gamma_2} z = 0 - 0 = 0$

$\forall \varphi \in A^*$ $\lambda \mapsto f(\lambda)\, \varphi((\lambda e - x)^{-1})$ is holomorphic on $\Omega \setminus \sigma(x)$ (by Prop. 8 (iv)) hence, by Cauchy theorem

$$\int_{\Gamma_1 \dot{-} \Gamma_2} f(\lambda)\, \varphi((\lambda e - x)^{-1})\, d\lambda = 0$$

But, further $\nearrow$ ||

$$\varphi\left(\int_{\Gamma_1 \dot{-} \Gamma_2} f(\lambda)(\lambda e - x)^{-1} d\lambda\right)$$

So, this holds for each $\varphi \in A^*$, so

$$0 = \int_{\Gamma_1 \dot{-} \Gamma_2} f(\lambda)(\lambda e - x)^{-1} d\lambda = \int_{\Gamma_1} f(\lambda)(\lambda e - x)^{-1} d\lambda - \int_{\Gamma_2} f(\lambda)(\lambda e - x)^{-1} d\lambda$$

$\lrcorner$

(3) $f \mapsto \tilde{f}(x)$ is linear. [This is clear]

(4) $f(\lambda) = \lambda^n$, where $n \in \mathbb{N}_0$. Then $\tilde{f}(x) = x^n$ (where $x^0 = e$)

$\ulcorner$ f is an entire function, so we can take $\Omega = \mathbb{C}$ and suppose that Γ is a circle with center 0 and radius $R > r(x)$ (positively oriented)

Since $(\lambda e - x)^{-1} = \sum_{k=0}^{\infty} \frac{x^k}{\lambda^{k+1}}$, $|\lambda| > r(x)$, we have, for any $\varphi \in A^*$:

$$\varphi(\tilde{f}(x)) = \frac{1}{2\pi i} \int_{\Gamma} \lambda^n \varphi\left(\sum_{k=0}^{\infty} \frac{x^k}{\lambda^{k+1}}\right) d\lambda = \frac{1}{2\pi i} \int_{\Gamma} \sum_{k=0}^{\infty} \frac{\varphi(x^k)}{\lambda^{k+1-n}}\, d\lambda =$$

$$= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(x^k)}{\lambda^{k+1-n}}\, d\lambda = \varphi(x^n),$$

as $\frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\lambda^{k+1-n}}\, d\lambda = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$

Hence $\tilde{f}(x) = x^n$ $\lrcorner$

(5) It follows that $\widetilde{id}(x)=x$, $\tilde{1}(x)=e$ and, if p is a polynomial, then $\tilde{p}(x)=p(x)$.
So, (b) and (c) are proved

(6) We prove (d), i.e. $\tilde{f}(\mu e)=f(\mu)e$ whenever $\mu\in\Omega$

$\ulcorner$ $\sigma(\mu e)=\{\mu\}$. Suppose that Γ is the circle centered at μ with positive orientation and radius $r>0$ s.t. $\overline{U(\mu,r)}\subset\Omega$.

Then $\tilde{f}(\mu e)=\frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda e-\mu e)^{-1}d\lambda=\frac{1}{2\pi i}\int_\Gamma \frac{f(\lambda)}{\lambda-\mu}\cdot e\, d\lambda$

$=f(\mu)e$ by the Cauchy formula $\lrcorner$

(7) Let $f\in H(\Omega)$, $\mu\in\mathbb{C}$, $g(\lambda)=(\mu-\lambda)f(\lambda)$.

Then $\tilde{g}(x)=(\mu e-x)\tilde{f}(x)$

$\ulcorner$ Let us first assume that $\mu\in\mathbb{C}\setminus\sigma(x)$.
Then $\mu\in\rho(x)$, so $(\mu e-x)^{-1}$ exists

Fix $\varphi\in A^*$. Define $\psi(y)=\varphi((\mu e-x)\cdot y)$, $y\in A$
Then $\psi\in A^*$, $\|\psi\|\le\|\varphi\|\cdot\|\mu e-x\|$

$\varphi(\tilde{g}(x))=\frac{1}{2\pi i}\int_\Gamma g(\lambda)\cdot\varphi((\lambda e-x)^{-1})\,d\lambda=$

$=\frac{1}{2\pi i}\int_\Gamma f(\lambda)\cdot(\mu-\lambda)\cdot\psi((\mu e-x)^{-1}(\lambda e-x)^{-1})\,d\lambda=$

$=\frac{1}{2\pi i}\int_\Gamma f(\lambda)\cdot\psi((\mu-\lambda)(\mu e-x)^{-1}(\lambda e-x)^{-1})\,d\lambda$

Prop 8(iii)

$=\frac{1}{2\pi i}\int_\Gamma f(\lambda)\cdot\psi((\lambda e-x)^{-1}-(\mu e-x)^{-1})\,d\lambda$

$$= \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)\, \varphi((\lambda e - x)^{-1})\, d\lambda - \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)\cdot \varphi((\mu e - x)^{-1})$$

$$= \varphi(\tilde{f}(x)) - \overset{\swarrow \text{ Cauchy theorem}}{0}$$

$$= \varphi((\mu e - x)\, \tilde{f}(x)).$$

$\varphi \in A^*$ arbitrary $\Rightarrow$ $\tilde{g}(x) = (\mu e - x)\, \tilde{f}(x)$.

Next, if $\mu \in \sigma(x)$, fix $\mu_0 \in \mathbb{C} \setminus \sigma(x)$

Then $$(\mu - \lambda) f(\lambda) = \underbrace{(\mu_0 - \lambda) f(\lambda)}_{g_1(\lambda)} + \underbrace{(\mu - \mu_0) f(\lambda)}_{g_2(\lambda)}$$

$\tilde{g}_1(x) = (\mu_0 e - x)\, \tilde{f}(x)$ by the first part

$\tilde{g}_2(x) = (\mu - \mu_0)\, \tilde{f}(x)$ by the linearity (see (3))

again using linearity, we see that

$$\tilde{g}(x) = \tilde{g}_1(x) + \tilde{g}_2(x) = (\mu e - x)\, \tilde{f}(x) \quad \lrcorner$$

(8) $f \in H(\Omega)$, $\mu \in \mathbb{C} \setminus \Omega$, $g(\lambda) = \frac{f(\lambda)}{\mu - \lambda}$ $\Rightarrow \tilde{g}(x) = (\mu e - x)^{-1} \tilde{f}(x)$

$\ulcorner$ by (7) we have $\tilde{f}(x) = (\mu e - x)\, \tilde{g}(x)$, $\mu \in \rho(x) \Rightarrow$

$\Rightarrow \tilde{g}(x) = (\mu e - x)^{-1} \tilde{f}(x)$ $\lrcorner$

(9) $f \in H(\Omega)$, p polynomial $\Rightarrow$ $\widetilde{p \cdot f}(x) = \tilde{p}(x) \cdot \tilde{f}(x)$

$\ulcorner$ By induction from (7) and using (c) $\lrcorner$

(10) $f(\lambda) = \frac{(\lambda - \xi_1) \cdots (\lambda - \xi_n)}{(\lambda - \theta_1) \cdots (\lambda - \theta_m)}$, $\xi_1 \dots \xi_n \in \mathbb{C}$, $\theta_1 \dots \theta_m \in \mathbb{C} \setminus \sigma(x)$

Then $\tilde{f}(x) = (x-\theta_1 e)^{-1} \dots (x-\theta_m e)^{-1}(x-\xi_1 e) \dots (x-\xi_n e)$

$\ulcorner$ By induction from ⑦ and ⑧ $\lrcorner$

(11) $f \in H(\Omega)$, $g \in H(\Omega)$, g a rational function

$\Rightarrow \widetilde{fg}(x) = \tilde{g}(x) \cdot \tilde{f}(x)$

$\ulcorner$ Using (10), ⑦, ⑧ and induction $\lrcorner$

(12) We prove (e): $f_n \overset{loc}{\rightrightarrows} f$ on Ω, $f_n \in H(\Omega)$

$\Rightarrow \tilde{f}_n(x) \to \tilde{f}(x)$ in the norm of A

$\ulcorner$ • $f \in H(\Omega)$ by Weierstrass theorem

• $\varphi \in A^*$, $\|\varphi\| \leq 1$. Then:

$$\| \varphi(\tilde{f}(x)) \| = \| \frac{1}{2\pi i} \int_\Gamma f(\lambda) \cdot \varphi((\lambda e - x)^{-1}) \, d\lambda \| \leq$$

$$\leq \frac{1}{2\pi} \cdot \text{length}(\Gamma) \cdot \max_{\lambda \in \langle\Gamma\rangle} \| f(\lambda) \cdot \varphi((\lambda e - x)^{-1}) \|$$

$$\leq \frac{1}{2\pi} \text{length}(\Gamma) \cdot \max_{\lambda \in \langle\Gamma\rangle} (|f(\lambda)| \cdot \|\varphi\| \cdot \|(\lambda e - x)^{-1}\|)$$

$$\leq \underbrace{\frac{1}{2\pi} \cdot \text{length}(\Gamma) \cdot \max_{\lambda \in \langle\Gamma\rangle} \|(\lambda e - x)^{-1}\|}_{C,\ \text{a constant not depending on } \varphi \text{ and } f} \cdot \max_{\lambda \in \langle\Gamma\rangle} |f(\lambda)|$$

• So, by H-B thm (Corollary II.8), we have

$$\| \tilde{f}(x) \| \leq C \cdot \max_{\lambda \in \langle\Gamma\rangle} |f(\lambda)|$$

Finally, if $f_n \overset{loc}{\rightrightarrows} f$ on Ω, then $f_n \rightrightarrows f$ on $\langle \Gamma \rangle$ (as $\langle \Gamma \rangle$ is compact),

so $f_n - f \rightrightarrows 0$ on $\langle \Gamma \rangle$, so

$\| \tilde{f_n}(x) - \tilde{f}(x) \| \to 0$ by the estimate. ┘

(13) $f, g \in H(\Omega) \Rightarrow \widetilde{fg}(x) = \tilde{f}(x) \cdot \tilde{g}(x)$

┌ Runge theorem $\Rightarrow \exists f_n \in H(\Omega)$ rational functions $f_n \overset{loc}{\rightrightarrows} f$ on Ω

Then $f_n g \overset{loc}{\rightrightarrows} fg$ on Ω. So,

$\widetilde{fg}(x) \overset{(12)}{=} \lim_{n\to\infty} \widetilde{f_n g}(x) \overset{(11)}{=} \lim_{n\to\infty} \tilde{f_n}(x) \tilde{g}(x) =$

$\overset{(12)}{=} \tilde{f}(x) \tilde{g}(x)$. ┘

(14) We have proved (a). It follows from (3) and (13) and (b)

(15) We prove (f): $\tilde{f}(x) \in G(A) \Leftrightarrow \forall \lambda \in \sigma(x) : f(\lambda) \neq 0$

┌ $\Leftarrow$: $\frac{1}{f}$ is holomorphic on an open set containing $\sigma(x)$

$1 = f \cdot \frac{1}{f} = \frac{1}{f} \cdot f$, so $e = \tilde{1}(x) = \tilde{f}(x) \cdot \widetilde{\frac{1}{f}}(x)$

$= \widetilde{\frac{1}{f}}(x) = \tilde{f}(x)$

So, $\widetilde{\frac{1}{f}}(x) = (\tilde{f}(x))^{-1}$

$\Rightarrow$ Suppose $\exists \lambda_0 \in \sigma(x) : f(\lambda_0) = 0$. Then $\exists g \in H(\Omega)$,
$f(\lambda) = (\lambda - \lambda_0) g(\lambda) \Rightarrow \tilde{f}(x) = (x - \lambda_0 e) \tilde{g}(x)$
$(x - \lambda_0 e)$ not invertible $\Rightarrow \tilde{f}(x)$ not invertible ┘

(16) We prove (g): $\sigma(\tilde{f}(x)) = f(\sigma(x))$

$\ulcorner$ $\lambda_0 \in \sigma(\tilde{f}(x)) \Leftrightarrow \lambda_0 e - \tilde{f}(x) \notin G(A)$
$$\lambda_0 e - \tilde{f}(x) = \widetilde{(\lambda_0 - f)}(x)$$

$\overset{(f)}{\Longleftrightarrow}$ $\exists\, \lambda \in \sigma(x)$: $(\lambda_0 - f)(\lambda) = 0$

$\Leftrightarrow \exists\, \lambda \in \sigma(x)$: $f(\lambda) = \lambda_0 \Leftrightarrow \lambda_0 \in f(\sigma(x))$ $\lrcorner$

(17) We prove (h): $f \in H(\Omega)$, $\Omega' \supset f(\sigma(x))$ open, $g \in H(\Omega')$

$\Rightarrow \widetilde{g \circ f}(x) = \tilde{g}(\tilde{f}(x))$

$\ulcorner$ By (g) we know $\sigma(\tilde{f}(x)) = f(\sigma(x))$. Let Γ_1 be a cycle in Ω' around $\sigma(\tilde{f}(x)) = f(\sigma(x))$.

$\Omega'_0 := \{\lambda \in \mathbb{C} \setminus \langle \Gamma_1 \rangle ;\ \operatorname{ind}_{\Gamma_1} \lambda = 1\}$. Then Ω'_0 is open, $\sigma(\tilde{f}(x)) \subset \Omega'_0 \subset \Omega'$.

Let $\Omega_0 = \{\lambda \in \Omega;\ f(\lambda) \in \Omega'_0\}$. Then Ω_0 is open (as Ω, Ω'_0 are open and f is cts),

$\sigma(x) \subset \Omega_0 \subset \Omega$ ← clear

$\lambda \in \sigma(x) \Rightarrow f(\lambda) \in f(\sigma(x)) = \sigma(\tilde{f}(x)) \subset \Omega'_0$

Let Γ_2 be a cycle in Ω_0 around $\sigma(x)$.

Then, given $\varphi \in A^*$, we have

$$\varphi(\tilde{g}(\tilde{f}(x))) = \frac{1}{2\pi i} \int_{\Gamma_1} g(\lambda) \cdot \varphi\left((\lambda e - \tilde{f}(x))^{-1}\right) d\lambda =$$

$$\overset{(a)}{=} \frac{1}{2\pi i} \int_{\Gamma_1} g(\lambda) \cdot \varphi\left(\widetilde{\left(\frac{1}{\lambda - f}\right)}(x) \right) d\lambda =$$

$$= \frac{1}{2\pi i} \int_{\Gamma_1} g(\lambda) \cdot \left(\frac{1}{2\pi i} \int_{\Gamma_2} \frac{1}{\lambda - f(\mu)} \cdot \varphi((\mu e - x)^{-1})\, d\mu \right) d\lambda =$$

$$\overset{\boxed{*}}{=} \frac{1}{2\pi i} \int_{\Gamma_2} \varphi((\mu e - x)^{-1}) \cdot \underbrace{\left(\frac{1}{2\pi i} \int_{\Gamma_1} \frac{g(\lambda)}{\lambda - f(\mu)}\, d\lambda \right)} d\mu =$$

$$= \frac{1}{2\pi i} \int_{\Gamma_2} \varphi((\mu e - x)^{-1})\, g(f(\mu))\, d\mu$$

$= g(f(\mu))$ by the Cauchy formula:
- g is holomorphic on Ω)
- $\mathrm{ind}_{\Gamma_1} z = 0$ for $z \in \mathbb{C} \setminus \Omega$)
- $\mathrm{ind}_{\Gamma_1} f(\mu) = 1$ as $f(\mu) \in \Omega_0'$

$$= \widetilde{g \circ f}(x)$$

$\boxed{*}$ Fubini theorem:

$$(\lambda, \mu) \mapsto \varphi((\mu e - x)^{-1}) \frac{g(\lambda)}{\lambda - f(\mu)} \quad \text{is cts on } \langle\Gamma_1\rangle \times \langle\Gamma_2\rangle$$

If we use the definition of path integral, we obtain a bounded measurable function on a product of two compact subsets of $\mathbb{R}$ (finite unions of intervals), so it is integrable.

(13) We prove (i): y commutes with $x \Rightarrow y$ commutes with $\tilde{f}(x)$

Γ $\varphi \in A^*$ ... define $\psi_1(z) = \varphi(yz)$, $z \in A$, $\psi_2(z) = \varphi(zy)$, $z \in A$
$\Rightarrow \psi_1, \psi_2 \in A^*$

$$\varphi(y \tilde{f}(x)) = \psi_1(\tilde{f}(x)) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)\, \psi_1((\lambda e - x)^{-1})\, d\lambda = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)\, \varphi(y(\lambda e - x)^{-1})\, d\lambda$$

$$= \frac{1}{2\pi i}\int_\Gamma f(\lambda)\,\varphi((\lambda e - x)^{-1}y)\,d\lambda = \frac{1}{2\pi i}\int_\Gamma f(\lambda)\,\psi_2((\lambda e - x)^{-1}) =$$

$$= \psi_2(\tilde f(x)) = \varphi(\tilde f(x)\,y)$$

[[$yx = yx \Rightarrow y(\lambda e - x) = (\lambda e - x)y \Rightarrow (\lambda e - x)^{-1}y = y(\lambda e - x)^{-1}$]]

Hence, $\forall \varphi \in A^*$: $\varphi(y\tilde f(x)) = \varphi(\tilde f(x)y)$. So, $y\tilde f(x) = \tilde f(x)y$.

(19) Remarks:

- It may happen that $f = g$ on $\sigma(x)$, but $\tilde f(x) \neq \tilde g(x)$

[Example: $A = M_2$, $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\sigma(x) = \{0\}$

$$\left.\begin{array}{l} f(\lambda) = \lambda \\ g(\lambda) = \lambda^2 \end{array}\right\} \Rightarrow f(0) = g(0) = 0.$$

So, $f = g$ on $\sigma(x)$, but

$$\left.\begin{array}{l} f(x) = x \\ g(x) = x^2 = 0 \end{array}\right\} f(x) \neq g(x)$$]

- $f \mapsto \tilde f$ need not be one-to-one, i.e.

$\tilde f(x) = \tilde g(x) \not\Rightarrow f = g$ on a nbhd of $\sigma(x)$

[As above, $A = M_2$, $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$f(\lambda) = 0$, $g(\lambda) = \lambda^2$. Then $\tilde f(x) = \tilde g(x) = 0$]

- $\tilde f(x) = \tilde g(x) \Rightarrow f = g$ on $\sigma(x)$

[$h = f - g \Rightarrow \tilde h(x) = 0 \Rightarrow h(\sigma(x)) = \sigma(\tilde h(x)) = \sigma(0) = \{0\}$

$\Rightarrow h = 0$ on $\sigma(x) \Rightarrow f = g$ on $\sigma(x)$.]

- Holomorphic calculus in non-unital algebras:

A B-algebra without unit, $x \in A$, $\Omega \subset \mathbb{C}$ open, $\sigma(x) \subset \Omega$.

Consider A^+ and identify A with $\tilde{A} = \{(a,0);\ a \in A\}$

Recall that $\sigma(x) = \sigma_{A^+}(x,0)$.

So, $\forall f \in H(\Omega)$ we may define $\tilde{f}(x) := \tilde{f}(x,0) \in A^+$

Further $\tilde{f}(x) \in \tilde{A} \iff f(0)=0$

Observe that $0 \in \sigma(x) \subset \Omega$, so $f(0)$ is defined

Further, define $\varphi \in (A^+)^*$ by $\varphi(a,t) = t$, $(a,t) \in A^+$

Then $\varphi(\tilde{f}(x)) = \frac{1}{2\pi i} \int_\Gamma f(\lambda) \cdot \varphi((\lambda e - x)^{-1})\, d\lambda =$

$= \frac{1}{2\pi i} \int_\Gamma f(\lambda)\ \underbrace{\varphi\left((-x, \lambda)^{-1}\right)}_{= \frac{1}{\lambda} \text{ by the definition of } A^+} = \frac{1}{2\pi i} \int_\Gamma \frac{f(\lambda)}{\lambda} = f(0)$

Cauchy formula

$0 \in \sigma(x) \subset \Omega$, $\operatorname{ind}_\Gamma 0 = 1$

In particular $\tilde{f}(x) \in \tilde{A} \iff \varphi(\tilde{f}(x)) = 0 \iff f(0) = 0$

Conclusion: The holomorphic calculus works also in algebras without unit, but only for functions satisfying $f(0)=0$.