Prop. X.24

Let A be a unital Banach algebra

(a) $I \subset A$ ideal of codimension 1 $\Rightarrow \exists!\, h \in \Delta(A)$ s.t. $I = \operatorname{Ker} h$

I is closed by Prop. 19

Take A/I and consider $q : A \to A/I$ the canonical quotient mapping

Then
- A/I is a Banach algebra of dimension 1
- q is a homomorphism.

A/I of dimension 1 $\Rightarrow$ $A/I = \{\lambda e_1,\ \lambda \in \mathbb{C}\} \simeq \mathbb{C}$

So, in fact $q \in \Delta(A)$

Uniqueness: $\operatorname{Ker} h_1 = \operatorname{Ker} h_2 \Rightarrow h_2$ is a multiple of h_1. (for example by Lemma VI.3)

But since $h_1(e) = h_2(e) = 1$, necessarily $h_1 = h_2$

(b) A commutative $\Rightarrow$ $h \mapsto \operatorname{Ker} h$ is a bijection of $\Delta(A)$ onto the set of maximal ideals in A.

$h \in \Delta(A) \Rightarrow \operatorname{Ker} h$ is an ideal of codimension 1

So, by (a)

$h \mapsto \operatorname{Ker} h$ is a bijection of $\Delta(A)$ onto the set of all the ideals of codimension 1.

It remains to prove that any maximal ideal is of codim. 1:

Let I be a closed ideal of codim ≥ 2. Then $B := A/I$ is a B-algebra of dimension ≥ 2. By Gelfand-Mazur (Theorem X.10) there is $x \in B$ not invertible. Let

$J = xB = \{xy\ ;\ y \in B\}$. $\Rightarrow J$ is an ideal in B, $J \neq \{0\}$

($x \in J$, $e \notin J$)

So, $q^{-1}(J)$ (where $q : A \to A/I = B$ is the canonical quotient map)

is an ideal in A strictly containing I. So, I is not maximal.