Prop. XI.5

(a) Let A be a Banach algebra with involution. Then A^+ is again a B-algebra with involution if we set
$$(a,\lambda)^* = (a^*, \bar{\lambda}) \quad , \quad (a,\lambda) \in A^+$$

⌈ The only nontrivial axiom is $((a,\lambda)(b,\mu))^* = (b,\mu)^*(a,\lambda)^*$

And this holds:
$$((a,\lambda)(b,\mu))^* = (ab+\lambda b+\mu a, \lambda\mu)^* = ((ab+\lambda b+\mu a)^*, \bar{\lambda}\bar{\mu}) =$$
$$= (b^*a^* + \bar{\lambda} b^* + \bar{\mu}\bar{a}, \bar{\lambda}\cdot\bar{\mu}) = (b^*, \bar{\mu})(a^*, \bar{\lambda}) =$$
$$= (b,\mu)^* \, (a,\lambda)^*. \quad ⌋$$

(b) If A is a C^*-algebra and we define
$$\|(a,\lambda)\| = \max\{|\lambda| ; \sup_{\substack{b\in A \\ \|b\|\leq 1}} \|ab+\lambda b\|\} \quad , \quad (a,\lambda)\in A^+ ,$$

then A^+ is a C^*-algebra

⌈ ① Set $p_1(a,\lambda)) := |\lambda| \quad , \quad (a,\lambda)\in A^+$

Then p_1 is a seminorm
$$p_1((a,\lambda)(b,\mu)) \leq p_1(a,\lambda)\, p_1(b,\mu) \qquad [\text{In fact }„="]$$
$$p_1((a,\lambda)^*(a,\lambda)) = p_1(a,\lambda)^2$$

... this is obvious from definitions ⌋

② Set $p_2(a,\lambda)) = \sup_{\substack{b\in A \\ \|b\|\leq 1}} \|ab+\lambda b\| \quad , \quad (a,\lambda)\in A^+$

Interpretation: define $L(a,\lambda))(b) = ab+\lambda b \quad , \quad b\in A$

Then $L(a,\lambda)) : A\to A$ linear, $\|L(a,\lambda))\| \leq \|a\|+|\lambda|$
So, $L(a,\lambda)) \in L(A)$.

Moreover, $p_2(a,\lambda)) = \|L(a,\lambda))\|$ by the definitions

③ $(a,\lambda) \mapsto L(a,\lambda)$ is linear (clear)

$L((a_1,\lambda_1)(a_2,\lambda_2)) = L_{(a_1,\lambda_1)} L_{(a_2,\lambda_2)}$

$\ulcorner$ $L_{(a_1,\lambda_1)} L_{(a_2,\lambda_2)}(b) = L_{(a_1,\lambda_1)}(a_2 b + \lambda_2 b) =$

$= a_1 a_2 b + \lambda_2 a_1 b + \lambda_1 a_2 b + \lambda_1 \lambda_2 b =$

$= L(a_1 a_2 + \lambda_2 a_1 + \lambda_1 a_2, \lambda_1 \lambda_2)(b) = L(((a_1,\lambda_1)(a_2,\lambda_2)))(b)$. $\lrcorner$

④ Thus p_2 is a seminorm and $p_2((a_1,\lambda_1)(a_2,\lambda_2)) \leq$
$\leq p_2(a_1,\lambda_1)\, p_2(a_2,\lambda_2)$

$\ulcorner$ This follows from ② and ③ $\lrcorner$

⑤ $p_2((a,\lambda)^* \cdot (a,\lambda)) = p_2((a,\lambda))^2$

$\ulcorner$ It is enough to prove "$\geq$"

$p_2((a,\lambda)^*(a,\lambda)) = p_2((a^*a + \bar{\lambda} a + \lambda a^*, \bar{\lambda}\cdot\lambda)) =$

$= \sup\limits_{\substack{b\in A \\ \|b\|\leq 1}} \|a^*ab + \bar{\lambda} ab + \lambda a^* b + \bar{\lambda}\lambda b\| \geq$

$\geq \sup\limits_{\|b\|\leq 1} \|b^*a^*ab + \bar{\lambda} b^* ab + \lambda b^* a^* b + \bar{\lambda}\lambda b^* b\| \geq$

$= \sup\limits_{\|b\|\leq 1} \|(b^*a^* + \bar{\lambda} b^*)(ab + \lambda b)\| = \sup\limits_{\|b\|\leq 1} \|(ab+\lambda b)^*(ab+\lambda b)\|$

$= \sup\limits_{\|b\|\leq 1} \|ab + \lambda b\|^2 = p_2(a,\lambda)^2$ $\lrcorner$

(6) $p_2(a,0) = \|a\|$ for $a \in A$

⌈ $p_2(a,0) = \sup\limits_{\substack{b\in A \\ \|b\|\le 1}} \|ab\| \le \sup\limits_{\substack{b\in A \\ \|b\|\le 1}} \|a\|\cdot\|b\| = \|a\|$

conversely: if $a=0$, the $p_2(a,0) = p_2(0,0) = 0$

if $a \neq 0$, then

$$p_2(a,0) \ge \left\| a \frac{a^*}{\|a\|} \right\| = \|a\|,$$

so the equality follows. ⌋

(7) $\|(a,\lambda)\| = \max\{p_1(a,\lambda), p_2(a,\lambda)\}$

$\Rightarrow$ $\|\cdot\|$ is a seminorm satisfying

$$\|(a_1,\lambda_1)(a_2,\lambda_2)\| \le \|(a_1,\lambda_1)\|\,\|(a_2,\lambda_2)\|$$
$$\|(a,\lambda)^*(a,\lambda)\| = \|(a,\lambda)\|^2$$

⌈ By (1), (4), (5) ⌋

(8) $\|\cdot\|$ is a norm:

⌈ $\|(a,\lambda)\| = 0 \Rightarrow |\lambda| = p_1(a,\lambda) = 0$, i.e. $\lambda = 0$

$p_2(a,0) = \|(a,0)\| = \|a\|$ (by (6))

so $a = 0$. ⌋

(9) $(A^+, \|\cdot\|)$ is a C^*-algebra

⌈ It is enough to show that $\|\cdot\|$ is complete

This follows, for example, from the fact that

$\|\cdot\|$ is equivalent to $\|\cdot\|_1$ defined by $\|(a,\lambda)\|_1 = \|a\| + |\lambda|$

To this end we prove $\|(a_n,\lambda_n)\|_1 \to 0 \iff \|(a_n,\lambda_n)\| \to 0$

$\Rightarrow$: $\|(a_n,\lambda_n)\|_1 \to 0 \Rightarrow \|a_n\| \to 0$ & $|\lambda_n| \to 0 \Rightarrow$ • $p_1(a_n,\lambda_n) = |\lambda_n| \to 0$

• $p_2(a_n,\lambda_n) \le \underbrace{p_2(a_n,0)}_{=\|a_n\| \text{ by (6)}} + \underbrace{p_2(0,\lambda_n)}_{=|\lambda_n| \text{ by definition of } p_2} = \|a_n\| + |\lambda_n| \to 0$

$\Rightarrow \|(a_n,\lambda_n)\| \to 0$

$\Leftarrow$: $\|(a_n,\lambda_n)\| \to 0 \Rightarrow \underbrace{p_1(a_n,\lambda_n)}_{=|\lambda_n|} \to 0$ & $p_2(a_n,\lambda_n) \to 0 \Rightarrow$ • $\lambda_n \to 0$

• $\|a_n\| \overset{(6)}{=} p_2(a_n,0) \le p_2(a_n,\lambda_n) + \underbrace{p_2(0,\lambda_n)}_{=|\lambda_n|} = p_2(a_n,\lambda_n) + |\lambda_n| \to 0$

$\Rightarrow \|(a_n,\lambda_n)\|_1 \to 0$ ⌋

(10) Suppose that A has no unit. Then p_2 is a norm

⌈ $p_2(a,\lambda)=0 \Rightarrow \forall b\in A \quad ab+\lambda b=0$

- $\lambda=0 \Rightarrow \forall b\in A: ab=0 \xRightarrow{\text{take } b=a^*} aa^*=0 \xRightarrow{\|aa^*\|=\|a\|^2} a=0$
- $\lambda\neq 0 \Rightarrow \forall b\in A: \ b=-\frac{a}{\lambda}\cdot b$

$\Rightarrow \quad -\frac{a}{\lambda}$ is a left unit, hence A is unital. ↯ ⌋

(11) If A has no unit, then (A^+,p_2) is a C^*-algebra, hence $p_2=\|\cdot\|$.

⌈ The equality follows from Corollary XI.4. So, it is enough to show that p_2 is complete.

Define $\theta: A^+\to\mathbb{C} \qquad \theta(a,\lambda)=\lambda$.

Then θ is a linear functional.

$\ker\theta=\{(a,0), a\in A\}$ ... it is closed as A is complete and p_2 is a norm

Thus θ is cts on (A^+,p_2)

We shall prove that p_2 is equivalent to $\|\cdot\|$, i.e.

$p_2(a_n,\lambda_n)\to 0 \Leftrightarrow \|(a_n,\lambda_n)\|\to 0$

$\Leftarrow$: clear, as $p_2\le\|\cdot\|$

$\Rightarrow$ Assume $p_2(a_n,\lambda_n)\to 0$. Then $\lambda_n=\theta(a_n,\lambda_n)\to 0$ (as θ is cts)

So, $p_1(a_n,\lambda_n)=|\lambda_n|\to 0$. It follows $\|(a_n,\lambda_n)\|\to 0$

⌋

(12) If A has a unit e, then p_2 is not a norm, for example $p_2(-e,1)=0$.

Then $p_2(a,\lambda)=\|a+\lambda e\|$, hence $\|(a,\lambda)\|=\max\{|\lambda|,\|a+\lambda e\|\}$