

Let  $A$  be a  $C^*$ -algebra and  $a \in A$

Prop. XI. 8

$$(a) \quad a^* = a \Rightarrow \sigma(a) \subset \mathbb{R}$$

Recall:  $\lambda \in \sigma(a) \Rightarrow |\lambda| \leq \|a\|$

If  $d + i\beta \in \sigma(a)$  ( $d, \beta \in \mathbb{R}$ ,  $\beta \neq 0$ ), then

$\forall r \in \mathbb{R} : d + i\beta + ir \in \sigma(a + ir e)$ , so

$$|d + i\beta + ir|^2 \leq \|a + ir e\|^2$$

$$d^2 + (\beta + r)^2$$

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$$d^2 + \beta^2 + 2\beta r + r^2$$

$$\|(a^* - ir e)(a + ir e)\| =$$

$$= \|a^* a + r^2 e\| \leq \|a^* a\| + r^2$$

So,  $2\beta r \leq \|a\|^2 - d^2 - \beta^2$  for all  $r \in \mathbb{R}$ .

It follows  $\beta = 0$ .  $\downarrow$

$$(5) \quad A \text{ unital, } a^* = a^{-1} \Rightarrow \sigma(a) \subset \mathbb{T} = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$$

$$\Gamma \quad a^* = a^{-1} \Rightarrow a^* a = a a^* = e$$

Hence  $\|a\|^2 = \|a^* a\| = \|e\| = 1$ , similarly  $\|a^*\| = 1$

$$\text{So, } \|a\| = \|a^*\| = 1$$

Thus  $\forall \lambda \in \sigma(a) : |\lambda| \leq 1$

and  $\forall \lambda \in \sigma(a^*) = \sigma(a^{-1}) : |\lambda| \leq 1$

Observe that  $\sigma(a^{-1}) = \left\{ \frac{1}{\lambda}, \lambda \in \sigma(a) \right\}$  whenever  $a$  is invertible

$\uparrow$  Indeed, the  $\lambda e - a = \frac{1}{\lambda} a (a^{-1} - \frac{e}{\lambda})$  for  $\lambda \neq 0$

and, since  $a$  is invertible, we deduce

$$\lambda e - a \in \mathcal{G}(A) \Leftrightarrow a^{-1} - \frac{e}{\lambda} \in \mathcal{G}(A) \quad \downarrow$$

So, necessarily  $\sigma(a) \subset \{\lambda, |\lambda| = 1\}$

(c)  $h \in \Delta(A) \Rightarrow h$  is a  $*$ -homomorphism

$$\Gamma \cdot a^* = a \stackrel{(a)}{\Rightarrow} \sigma(a) \subset \mathbb{R} \stackrel{\text{Prop X.22(e)}}{\Rightarrow} \overline{h(a)} = h(a) \in \mathbb{R}$$

$\ast$  a general  $\Rightarrow a = b + ic$ ,  $b, c$  selfadjoint

$$\text{Then } h(a^*) = h(b - ic) = h(b) - i h(c) = \overline{h(b) + i h(c)}$$

$$= \overline{h(b + ic)} = \overline{h(a)} \quad \square$$