

Proposition XI.12

Let A be a C^* -algebra and $B \subset A$ its G^* -subalgebra

(a) $\forall x \in B: \sigma_B(x) \cup \{0\} = \sigma_A(x) \cup \{0\}$

(b) A unital, $e \in B \Rightarrow \forall x \in B: \sigma_B(x) = \sigma_A(x)$.

In particular $G(B) = B \cap G(A)$

Proof (b) ① $x \in B, x^* = x \Rightarrow \sigma_A(x) \subset \mathbb{R} \Rightarrow \sigma_A(x)$ is connected
 \uparrow
 Prop. XI.8(c)

$\Rightarrow \sigma_A(x) = \sigma_B(x)$ by Prop. XI.15(c)

② $x \in B \cap G(A) \dots y := x^{-1}$

$\Rightarrow x^* \in B \cap G(A), (x^*)^{-1} = y^*$

Further, $x^*x \in B \cap G(A), (x^*x)^{-1} = yy^*$

Since x^*x is selfadjoint, $\sigma_A(x^*x) = \sigma_B(x^*x)$,
 in part. $0 \notin \sigma_B(x^*x)$, so $x^*x \in G(B)$,
 i.e. $yy^* \in B$

Finally, $y = yy^* \cdot x^* \in B$, so $x \in G(B)$

③ $x \in B, \lambda \in \mathbb{C}$

By ② $\lambda e - x \in G(B) \Leftrightarrow \lambda e - x \in G(A)$

so, $\sigma_A(x) = \sigma_B(x)$

(a) consider A^+ and let $\tilde{B} = \{(s, \lambda) | s \in B\} \subset A^+$

Then for $x \in B$ we have

$\sigma_{A^+}(x, 0) = \sigma_{\tilde{B}}(x, 0)$

\parallel $\sigma_A(x) \cup \{0\} = \sigma_B(x) \cup \{0\}$