Proposition XI.20 H .. a Hilbert space, $P \in \mathcal{L}(H)$ a projection, i.e. $P^2 = P$

(ii) $\Rightarrow$ (iii) P self-adjoint $\Rightarrow$ P normal
[trivial]

(iii) $\Rightarrow$ (i) P normal $\Rightarrow$ P orthogonal (i.e. $\ker P \perp R(P)$)

$\lceil$ $x \in \ker P, y \in R(P) \Rightarrow$ • $\|P^*x\|^2 = \langle P^*x, P^*x\rangle = \langle x, PP^*x\rangle = \langle x, P^*Px\rangle = \langle Px, Px\rangle = 0 \Rightarrow P^*x = 0$

• $\langle x, y\rangle = \langle x, Py\rangle = \langle \underbrace{P^*x}_{=0}, y\rangle = 0$ $\rfloor$

(i) $\Rightarrow$ (iv) P orthogonal $\Rightarrow$ $\langle Px, x\rangle = \|Px\|^2$, $x \in H$

$\lceil \langle Px, x\rangle = \langle Px, Px\rangle + \langle \underbrace{Px}_{\in R(P)}, \underbrace{x - Px}_{\in \ker P}\rangle = \langle Px, Px\rangle \rfloor$

(i) $\Rightarrow$ (ii) P orthogonal $\Rightarrow$ P self-adjoint

$\lceil$ $x, y \in H \Rightarrow$

$$\langle Px, y\rangle = \langle \underbrace{Px}_{\in R(P)}, \underbrace{y - Py}_{\in \ker P}\rangle + \langle Px, Py\rangle = \langle Px, Py\rangle$$

$$\langle x, Py\rangle = \langle \underbrace{x - Px}_{\in \ker P}, \underbrace{Py}_{\in R(P)}\rangle + \langle Px, Py\rangle = \langle Px, Py\rangle$$

$\Rightarrow \langle Px, y\rangle = \langle x, Py\rangle$. Hence, $P^* = P$ $\rfloor$

(iv) $\Rightarrow$ (v) $\forall x \in H: \langle Px, x\rangle = \|Px\|^2 \Rightarrow \|P\| \leq 1$

$\lceil$ $x \in B_H \Rightarrow \|Px\|^2 = \langle Px, x\rangle \leq \|Px\| \cdot \|x\| \leq \|P\|$

So, $\|P\|^2 \leq \|P\|$. Hence $\|P\| \leq 1$ $\rfloor$

(v) $\Rightarrow$ (i) $\quad \|P\| \leq 1 \Rightarrow P$ orthogonal

- $(\mathrm{Ker}\, P)^{\perp} \subset R(P)$

$\ulcorner$ $x \in (\mathrm{Ker}\, P)^{\perp}$. Then $x - Px \in \mathrm{Ker}\, P$, so

$\langle x - Px, x \rangle = 0$, hence $\langle x, x \rangle = \langle Px, x \rangle$

Thus $\|x\|^2 = \langle Px, x \rangle \leq \|Px\| \cdot \|x\| \leq \|x\|^2$

So, we have equalities. Thus $\|Px\| = \|x\|$ and Px is a multiple of x

$Px = \alpha x$. But $\langle x, x \rangle = \langle Px, x \rangle = \langle \alpha x, x \rangle =$
$= \alpha \langle x, x \rangle$, hence necessarily $\alpha = 1$ (or $x = 0$, but this is a trivial case)

Hence $Px = x$, so $x \in R(P)$. $\lrcorner$

- $R(P) \subset (\mathrm{Ker}\, P)^{\perp}$

$\ulcorner$ $x \in R(P) \Rightarrow x = x_1 + x_2$, $x_1 \in \mathrm{Ker}\, P$, $x_2 \in (\mathrm{Ker}\, P)^{\perp}$
By above $x_2 \in R(P)$.

Thus $x = Px = Px_1 + Px_2 = Px_2 = x_2 \in (\mathrm{Ker}\, P)^{\perp}$ $\lrcorner$

Moreover, if P, Q are OG projections, then $R(P) \perp R(Q) \Leftrightarrow PQ = 0$

$\ulcorner$ $R(P) \perp R(Q) \Leftrightarrow R(Q) \subset R(P)^{\perp} = \mathrm{Ker}\, P \Leftrightarrow PQ = 0$ $\lrcorner$