Def: A C^*-algebra, $x \in A$ is a partial isometry if x^*x and xx^* are projections

Let $x \in A$. TFAE:

(i) x is a partial isometry
(ii) x^*x is a projection
(iii) xx^* is a projection
(iv) $x = xx^*x$

Proof: (i) $\Rightarrow$ (ii) & (iii) trivial

(iv) $\Rightarrow$ (ii) Assume $x = xx^*x$. Then
$x^*x = x^*xx^*x = (x^*x)^2$. Since x^*x is self-adjoint, it is a projection.
Similarly $xx^* = xx^*xx^* = (xx^*)^2$. Since xx^* is self-adjoint, it is a projection.

(ii) $\Rightarrow$ (iv) WLOG A unital. Assume $p = x^*x$ is a projection.
Then $e-p$ is also a projection [$(e-p)^* = e-p$, $(e-p)^2 = e^2 - 2p + p^2 = e-p$]
and $x(e-p) = 0$
[$\|x(e-p)\|^2 = \|(e-p)x^*x(e-p)\| = \|(e-p)p(e-p)\| = 0$]
So, $x = xe - xp = xx^*x$

(iii) $\Rightarrow$ (iv) analogous: $q = xx^*$ is a projection $\Rightarrow$ $e-q$ is a projection
$\|(e-q)x\|^2 = \|x^*(e-q)\|^2 = \|(e-q)xx^*(e-q)\| = \|(e-q)q(e-q)\| = 0$
$\Rightarrow x = ex = qx = xx^*x$

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Fact: $T \in \mathcal{L}(H)$ is a partial isometry $\iff$ $\exists\, Y \subset H$ closed : $T|_Y$ is an isometry & $T|_{Y^\perp} = 0$

(Then T^*T is the OG projection onto Y, TT^* is the OG projection onto $R(T)$)

Proof: $\Rightarrow$: Assume $P = T^*T$ and $Q = TT^*$ are projections

Let $Y = R(P)$

For $x \in Y$ we have $\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^*Tx \rangle =$

$= \langle x, Px \rangle = \langle x, x \rangle = \|x\|^2$

So, $T|_Y$ is an isometry

Let $x \in Y^\perp = \ker P$, Then $\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^*Tx \rangle =$

$= \langle x, Px \rangle = \langle x, 0 \rangle = 0$

So, $T|_{Y^\perp} = 0$

Moreover: $R(T) = R(Q)$

$\supset$: clear, as $Q = TT^*$

$\subset$: Assume $x \in R(T) \Rightarrow \exists y \in H : x = Ty$.

Recall that $T = TT^*T$, so

$Qx = TT^*x = TT^*Ty = Ty = x$

$\Leftarrow$ Assume $T|_Y$ is an isometry, $T|_{Y^\perp} = 0$. Let P denote the OG projection onto Y. Let $Z = R(T)$

Then • $T^*|_Z = (T|_Y)^*$ $\left[y \in Y, z \in Z \Rightarrow \langle Ty, z \rangle = \langle y, T^*z \rangle \right]$

($T|_Y$ considered as $\in L(Y, Z)$)

• $(T|_Y)^* = (T|_Y)^{-1}$ by Prop. XI.18, (ii) $\Rightarrow$ (i)

• $T = TP$ by the assumptions.

So $T^*T = PT^*TP = P \underbrace{T^*|_Z \, T|_Y}_{= Id_Y} P = P$