Proposition XII.4 $\quad H$ Hilbert space, $T \in \mathcal{L}(H)$

- T normal $\iff \forall x \in H : \|Tx\| = \|T^*x\| \qquad (\square)$

$\ulcorner$

$$\|Tx\| = \|T^*x\| \iff \underbrace{\langle Tx, Tx\rangle}_{\langle T^*Tx, x\rangle} = \underbrace{\langle T^*x, T^*x\rangle}_{\langle TT^*x, x\rangle}$$

So $\forall x \in H \; \|Tx\| = \|T^*x\| \iff \forall x \in H : \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle$

$$\Updownarrow \text{ Prop. 3(c)}$$

$$T^*T = TT^* \qquad \lrcorner$$

Next suppose that T is normal

(a) $\operatorname{Ker} T = \operatorname{Ker} T^* = \mathcal{R}(T)^{\perp}$

$\ulcorner \; x \in \operatorname{Ker} T \iff Tx = 0 \iff \|Tx\| = 0 \overset{(\square)}{\iff} \|T^*x\| = 0 \iff T^*x = 0 \iff x \in \operatorname{Ker} T^*$

So, $\operatorname{Ker} T = \operatorname{Ker} T^*$

Further, $\operatorname{Ker} T^* = \mathcal{R}(T)^{\perp}$ for any operator T (including normal)

$\subset$: $x \in \operatorname{Ker} T^*$, $y \in \mathcal{R}(T)$. Fix $z \in H$ s.t. $y = Tz$

$\langle x, y\rangle = \langle x, Tz\rangle = \langle T^*x, z\rangle = \langle 0, z\rangle = 0$

So $x \in \mathcal{R}(T)^{\perp}$

$\supset$: $x \in \mathcal{R}(T)^{\perp} \Rightarrow \forall y \in H : \langle x, Ty\rangle = 0$

$$\langle x, Ty\rangle = \langle T^*x, y\rangle$$

$\iff T^*x = 0 \Rightarrow x \in \operatorname{Ker} T^* \qquad \lrcorner$

(b) $\overline{R(T)} = H \iff T$ is one-to-one. Hence $\sigma_r(T) = \emptyset$, $\sigma(T) = \sigma_{ap}(T)$

$\ulcorner\ \overline{R(T)} = H \iff R(T)^\perp = \{0\} \overset{(a)}{\iff} \ker T = \{0\} \iff T$ is one-to-one

Further, $\lambda \in \mathbb{C} \Rightarrow \lambda I - T$ is normal

so $(\lambda I - T)$ is one-to-one $\iff R(\lambda I - T)$ is dense.

Thus $\sigma_r(T) = \emptyset$ and, by Prop. 1(c), $\sigma(T) = \sigma_{ap}(T)$ $\lrcorner$

(c) $\lambda \in \mathbb{C}$, $x \in H$. Then $Tx = \lambda x \iff T^* x = \bar{\lambda} x$

In particular $\sigma_p(T^*) = \{\bar{\lambda};\ \lambda \in \sigma_p(T)\}$

$\ulcorner\ \lambda \in \mathbb{C} \Rightarrow \lambda I - T$ is normal, $(\lambda I - T)^* = \bar{\lambda} I - T^*$

Thus $\ker(\lambda I - T) = \ker(\bar{\lambda} I - T^*)$ by (a) $\lrcorner$

(d) $\lambda_1, \lambda_2 \in \sigma_p(T)$, $\lambda_1 \neq \lambda_2 \Rightarrow \ker(\lambda_1 I - T) \perp \ker(\lambda_2 I - T)$

$\ulcorner\ x \in \ker(\lambda_1 I - T)$, $y \in \ker(\lambda_2 I - T)$

Then $(\lambda_1 - \lambda_2)\langle x, y\rangle = \langle \lambda_1 x, y\rangle - \langle x, \bar{\lambda}_2 y\rangle =$

$\overset{(c)}{=} \langle Tx, y\rangle - \langle x, T^* y\rangle = \langle Tx, y\rangle - \langle Tx, y\rangle = 0$

so $\langle x, y\rangle = 0$. $\lrcorner$