Theorem XII.6 (polar decomposition)

- $T \in \mathcal{L}(H) \Rightarrow T^*T$ is positive

$$\ulcorner \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 \geq 0 \lrcorner$$

- $T \in \mathcal{L}(H) \Rightarrow |T| := \sqrt{T^*T}$ $\quad$ (T^*T self-adjoint, $\sigma(T^*T) \subset [0,\infty)$)

- $x \in H \Rightarrow$ $\quad \|\,|T|x\|^2 = \langle |T|x, |T|x\rangle = \langle |T|^2 x, x\rangle =$

$$= \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2$$

So, define $U \in \mathcal{L}(H)$ as follows:

- for $y = |T|x \in R(|T|)$ define $Uy := Tx$

as $\|\,|T|x\| = \|Tx\|$, U is an isometry of $R(|T|)$ onto $R(T)$

So, it can be extended to $\overline{R(|T|)}$

- for $y \in R(|T|)^\perp$ put $Uy = 0$

Then U is a partial isometry, $U|T| = T$, $U\restriction_{R(|T|)^\perp} = 0$.

The uniqueness is clear.

- Similarly define a partial isometry V s.t. $|T| = VT$ and $V\restriction_{R(T)^\perp} = 0$.

Then $U^* = V$: $x, y \in H$ $\quad x = x_1 + x_2$ $(x_1 \in \overline{R(T)}, x_2 \in R(T)^\perp)$, $y = y_1 + y_2$ $(y_1 \in \overline{R(|T|)}, y_2 \in R(|T|)^\perp)$

$\langle Vx, y\rangle = \langle Vx_1, y_1 + y_2\rangle = \langle Vx_1, y_1\rangle$ as $Vx_1 \in \overline{R(|T|)}$

$\langle x, Uy\rangle = \langle x_1 + x_2, Uy_1\rangle = \langle x_1, Uy_1\rangle$ as $Uy_1 \in \overline{R(T)}$

Recall $U(|T|x) = Tx$, $V(Tx) = |T|x$, $x \in H$

Thus $U_{\restriction R(|T|)} = (V_{\restriction R(T)})^{-1}$

This can be extended to the closures:

$U_{\restriction \overline{R(|T|)}} = (V_{\restriction \overline{R(T)}})^{-1}$.

Hence $\langle Vx_1, y_1 \rangle \underset{\Uparrow}{=} \langle UVx_1, Uy_1 \rangle = \langle x_1, Uy_1 \rangle$

$U_{\restriction \overline{R(|T|)}}$ is an isometry, use Prop XI.18

Thus $V = U^*$.