X.3 Holomorphic functional calculus

Proposition 17 (path integral with values in a Banach space). *Let $\varphi : [a,b] \to \mathbb{C}$ be a continuous piecewise $\mathcal{C}^1$-smooth curve (i.e., φ is a continuous mapping and there exists a partition of the interval $[a,b]$ such that on each of its intervals the derivative φ' is continuous and has finite unilateral limits at the endpoints). Let X be a Banach space and let $f : \langle\varphi\rangle \to X$ be a continuous mapping (where $\langle\varphi\rangle = \varphi([a,b])$ is the range of φ). Then the integral*

$$\int_\varphi f = \int_a^b f(\varphi(t))\varphi'(t)\,\mathrm{d}t$$

exists in the Bochner sense.

Remarks:

(1) Similarly as in the complex analysis, we will also consider the integral over a cycle (i.e., over a formal sum of closed piecewise $\mathcal{C}^1$-smooth curves).

(2) To compute the path integral and to work with it we will use the weak version of the integral, i.e., the equivalence

$$x = \int_\varphi f \Leftrightarrow \forall x^* \in X^* : x^*(x) = \int_\varphi x^* \circ f.$$

(3) To define the above-mentioned path integral and to work with it the Bochner theory is not necessary. The integral exists also in the Riemann sense. Where the Riemann integral of a function $g : [a,b] \to X$ equals $x \in X$ if and only if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall a = t_0 < t_1 < \cdots < t_k = b \text{ partition of } [a,b] :$$

$$\max_{1\le j\le k}(t_j - t_{j-1}) < \delta \Rightarrow \forall u_1 \in [t_0,t_1],\ldots,u_k \in [t_{k-1},t_k] : \left\| x - \sum_{j=1}^k g(u_j)(t_j - t_{j-1}) \right\| < \varepsilon.$$

One can show that in our case the Riemann integral exists and, moreover, the equivalence from the preceding remark holds true. This approach can be found in the literature.

Definition. Let A be a Banach algebra with a unit e, let $x \in A$ and let f be a function holomorphic on an open set $\Omega \subset \mathbb{C}$ containing $\sigma(x)$. Let Γ be a „cycle around $\sigma(x)$ in Ω“(i.e., Γ is a cycle in $\Omega \setminus \sigma(x)$, $\operatorname{ind}_\Gamma z$ assumes only the values 0 or 1, $\operatorname{ind}_\Gamma z = 1$ for $z \in \sigma(x)$ and $\operatorname{ind}_\Gamma z = 0$ for $z \in \mathbb{C} \setminus \Omega$). We define the element $\tilde{f}(x) \in A$ by the formula

$$\tilde{f}(x) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda e - x)^{-1}\,d\lambda.$$

Remarks:

1. (1) We know from complex analysis that a cycle with the required properties exists.
2. (2) The element $\tilde{f}(x)$ is well defined due to Proposition 17.
3. (3) It follows by the Cauchy theorem that the value $\tilde{f}(x)$ does not depend on a concrete choice of the cycle Γ.
4. (4) The mappping $f \mapsto \tilde{f}(x)$ is called the **holomorphic functional calculus**, or the **Dunford functional calculus**.
5. (5) Instead of $\tilde{f}(x)$ one often writes just $f(x)$.

Theorem 18 (properties of the holomorphic calculus). *Let A be a Banach algebra with a unit e, let $x \in A$ and let $\Omega \subset \mathbb{C}$ be an open set containing $\sigma(x)$.*

- (a) *The mapping $f \mapsto \tilde{f}(x)$ is an algebraic homomorphism of the unital (commutative) algebra $H(\Omega)$ into A.*
- (b) $\widetilde{id}(x) = x$ *and* $\tilde{1}(x) = e$, *where* $id(\lambda) = \lambda$ *and* $1(\lambda) = 1$ *for* $\lambda \in \Omega$.
- (c) *If p is a polynomial, then $\tilde{p}(x) = p(x)$ where $p(x)$ has the meaning from the previous section.*
- (d) *If $\lambda \in \Omega$, then $\tilde{f}(\lambda e) = f(\lambda)e$.*
- (e) *If $f_n \to f$ locally uniformly on Ω (were $f_n \in H(\Omega)$ for each $n \in \mathbb{N}$), then $\tilde{f}_n(x) \to \tilde{f}(x)$ in A.*
- (f) *$\tilde{f}(x) \in G(A)$ if and only if $f(\lambda) \neq 0$ for each $\lambda \in \sigma(x)$.*
- (g) $\sigma(\tilde{f}(x)) = f(\sigma(x))$.
- (h) $\widetilde{(g \circ f)}(x) = \tilde{g}(\tilde{f}(x))$ *whenever* $f \in H(\Omega)$, $g \in H(\Omega')$, $\Omega' \supset f(\sigma(x))$.
- (i) *If $y \in A$ commutes with x (i.e., $xy = yx$), then y commutes with $\tilde{f}(x)$ for each $f \in H(\Omega)$.*

Remarks. Let A be a unital Banach algebra and let $x \in A$.

1. (1) If f and g are two holomorphic functions on a neighborhood of $\sigma(x)$, which coincide on a neighborhood of $\sigma(x)$, then $\tilde{f}(x) = \tilde{g}(x)$.
2. (2) It may happen that f and g coincide on $\sigma(x)$ and $\tilde{f}(x) \neq \tilde{g}(x)$.
3. (3) The assignment $f \mapsto \tilde{f}(x)$ need not be one-to-one. I.e., the equality $\tilde{f}(x) = \tilde{g}(x)$ does not imply that $f = g$ on a neighborhood of $\sigma(x)$.
4. (4) If $\tilde{f}(x) = \tilde{g}(x)$, then $f|_{\sigma(x)} = g|_{\sigma(x)}$.
5. (5) The holomorphic calculus may be defined also in non-unital algebras: Let A be an non-unital algebra, let $x \in A$ and let $\Omega \subset \mathbb{C}$ be an open set containing $\sigma(x)$. Consider the algebra A^+ and identify A with the subalgebra $\{(a, 0); a \in A\} \subset A^+$. Then, given $f \in H(\Omega)$, we may define $\tilde{f}(x) = \tilde{f}(x, 0) \in A^+$. Further, $\tilde{f}(x) \in A$ if and only if $f(0) = 0$ (recall that $0 \in \sigma(x) \subset \Omega$).