XI.2 $*$-homomorphisms and Gelfand transform of C*-algebras

Definition. Let A and B be C^*-algebras and let $h : B \to A$. We say that h is a $*$-**homomorphism**, if it is a homomorphism of Banach algebras satisfying moreover $h(x^*) = h(x)^*$ for each $x \in B$.

Proposition 6 (on the automatic continuity of a $*$-homomorphism). *Let A and B be C^*-algebras and let $h : B \to A$ be a $*$-homomorphism of B into A. Then $\|h\| \le 1$.*

Example 7. *Let K, L be compact Hausdorff spaces and let $\varphi : \mathcal{C}(K) \to \mathcal{C}(L)$ be a $*$-homomorphism satisfying $\varphi(1) = 1$. Then there is a continuous mapping $\alpha : L \to K$ such that $\varphi(f) = f \circ \alpha$ for $f \in \mathcal{C}(K)$. If φ is moreover one-to-one, then $\alpha(L) = K$, so φ is an isometry of $\mathcal{C}(K)$ into $\mathcal{C}(L)$.*

Proposition 8. *Let A be a C^*-algebra and let $a \in A$. Then:*

(a) *If a is selfadjoint, then $\sigma(a) \subset \mathbb{R}$.*
(b) *If A is unital and $a^* = a^{-1}$ (i.e., a is* **unitary***), then*

$$\sigma(a) \subset \mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

(c) *For $h \in \Delta(A)$ and $a \in A$ one has $h(a^*) = \overline{h(a)}$ (i.e., h is a $*$-homomorphism whenever it is a homomorphism).*

Theorem 9 (Gelfand-Naimark). *Let A be a commutative C^*-algebra and let $\Gamma : A \to \mathcal{C}_0(\Delta(A))$ be its Gelfand transform. Then Γ is an isometric $*$-isomorphism of the C^*-algebra A onto the C^*-algebra $C_0(\Delta(A))$ (in particular $\widehat{x^*} = \overline{\hat{x}}$ for $x \in A$).*

Therefore, A is unital if and only if $\Delta(A)$ is compact.

Corollary 10. *Let A and B be commutative C^*-algebras. Then A and B are $*$-isomorphic if and only if $\Delta(A)$ and $\Delta(B)$ are homeomorphic.*

Corollary 11. *Let A and B be C^*-algebras and let $h : B \to A$ be a one-to-one $*$-homomorphism of B into A. Then h is an isometry of B into A.*