XI.4 Distinguished elements of C*-algebras

Remark: In this section we define several distinguished types of elements of C^*-algebras. We will show their role in basic examples of C^*-algebras. We will use the following notation:

- A ... a general C^*-algebra;
- H ... a complex Hilbert spaces;
- $L(H)$ ... the C^*-algebra of bounded linear operators on H (with the operator norm, the operation of composition a the involution defined as the adjoint operator);
- Ω ... a Hausdorff locally compact space;
- $C_0(\Omega)$ ... the C^*-algebra of continuous functions on Ω with limit 0 at infinity (with the supremum norm, the operation of multiplication and the involution defined as complex conjugation).

Reminder: An element $x \in A$ is **selfadjoint** if $x^* = x$. An element $x \in A$ is **normal** if $x^*x = xx^*$.

Proposition 16.

(a) *A function $f \in C_0(\Omega)$ is selfadjoint if and only if it is real-valued. Any function $f \in C_0(\Omega)$ is normal.*
(b) *An operator $T \in L(H)$ is selfadjoint if and only if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for every $x, y \in H$.*

Reminder: Assume that A has a unit e. An element $x \in A$ is called **unitary** if $x^*x = xx^* = e$.

Proposition 17.

(a) *Any unitary element is normal.*
(b) *The algebra $C_0(\Omega)$ admits a unitary element if and only if Ω is compact. A function $f \in C_0(\Omega)$ is unitary if and only if $|f(t)| = 1$ for each $t \in \Omega$.*
(c) *An operator $T \in L(H)$ is unitary if and only if T is an isometry of H onto H.*

Proposition 18 (a characterization of unitary operators). *Let H and K be Hilbert spaces and $T \in L(H, K)$. Consider the following assertions:*

(i) *T is unitary (i.e., $T^*T = I_H$ and $TT^* = I_K$).*
(ii) *T is an isometry of H onto K.*
(iii) *T is an isometry of H into K.*
(iv) *$\langle Tx, Ty\rangle_K = \langle x, y\rangle_H$ for $x, y \in H$.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv). If T is assumed to be surjective, all the assertions are equivalent.

Definition.

- An element $x \in A$ is said to be a **projection** if $x^* = x = x^2$.
- Two projection $x, y \in A$ are called **mutually orthogonal** if $xy = 0$.

Proposition 19.

(a) *A function $f \in C_0(\Omega)$ is a projection if and only if it attains only values 0 and 1, i.e., if and only if $f = \chi_U$ where $U \subset \Omega$ an open compact subset.*

(b) *Two projection $\chi_U, \chi_V \in C_0(\Omega)$ are mutually orthogonal if and only if $U \cap V = \emptyset$.*

Proposition 20 (characterization of orthogonal projections). *Let H be a Hilbert space and let $P \in L(H)$ satisfy $P^2 = P$ (i.e., P is a projection as a linear operator). The following assertions are equivalent:*

(i) *P is an* **orthogonal projection**, *i.e.,* $\ker P \perp R(P)$.

(ii) *P is self-adjoint (i.e., P is a projection as an element of the C^*-algebra $L(H)$).*

(iii) *P is normal.*

(iv) $\langle Px, x \rangle = \|Px\|^2$ *for $x \in H$.*

(v) $\|P\| \le 1$.

Moreover, if $P, Q \in L(H)$ are two orthogonal projections, then $R(P) \perp R(Q)$ if and only if $PQ = 0$. In this case P and Q are called **mutually orthogonal**.

Remark. One more equivalent condition may be added to Proposition 20:

(vi) $\langle Px, x \rangle \ge 0$ for $x \in H$.

Implication (iv)⇒(vi) is obvious, implication (vi)⇒(ii) follows from Proposition XII.5(a) below.

Definition. An element $x \in A$ is called a **partial isometry** if elements x^*x and xx^* are projections (possibly different).

Proposition 21.

(a) *Let $x \in A$. Then:*

$$x \text{ is a partial isometry} \iff x^*x \text{ is a projection}$$
$$\iff xx^* \text{ is a projection} \iff xx^*\ x = xx^*x$$

(b) *A function $f \in C_0(\Omega)$ is a partial isometry if and only if $|f|$ attains only values 0 and 1, i.e., if and only if there exists $U \subset \Omega$ compact open such that $|f| = 1$ on U and $f = 0$ on $\Omega \setminus U$.*

(c) *An operator $T \in L(H)$ is a partial isometry if and only if there exists a closed subspace $Y \subset\subset H$ such that $T|_Y$ is an isometry (of Y into H) and $T|_{Y^\perp} = 0$.*