FUNCTIONAL ANALYSIS 2

SUMMER SEMESTER 2023/2024

PROBLEMS TO CHAPTER XII

Problems to Section XII.1 – more on bounded operators

Problem 1. Let H be a Hilbert space and let $U \in L(H)$.

(1) Assume that U^*U is an orthogonal projection. Show that U is a partial isometry.
(2) Assume that UU^* is an orthogonal projection. Show that U is a partial isometry.

`Hint:` *Analyze the polar decomposition of U (see Theorem XII.6).*

Problem 2. Let X be a (complex) Banach space and let $T \in L(X)$.

(1) Show that $\overline{\sigma_p(T)} \subset \sigma_{ap}(T)$.
(2) Show that $\lambda \in \mathbb{C} \setminus \sigma_{ap}(T)$ if and only if the operator $\lambda I - T$ is bounded from below (i.e., the function $x \mapsto \|(\lambda I - T)x\|$ is bounded from below on the unit sphere).
(3) Show that operators which are bounded below form an open set in $L(X)$.
(4) Deduce that $\sigma_{ap}(T)$ is a closed set (and provide an alternative proof of (1)).
(5) Show that $\partial\sigma(T) \subset \sigma_{ap}(T)$.

`Hint:` *(5) Let $\lambda \in \partial\sigma(T)$. Then there is a sequence (λ_n) in $\rho(T)$ converging to λ. By Theorem X.7(3) we get $\|(\lambda_n I - T)^{-1}\| \to \infty$, hence there are $x_n \in S_X$ with $\|(\lambda_n I - T)^{-1}x_n\| \to \infty$. Show that $y_n = \frac{(\lambda_n I - T)^{-1}x_n}{\|(\lambda_n I - T)^{-1}x_n\|}$ witness that $\lambda \in \sigma_{ap}(T)$.*

Problem 3. Let H be a (complex) Hilbert space and let $T \in L(H)$.

(1) Show that $\sigma(T^*) = \{\overline{\lambda}; \lambda \in \sigma(T)\}$.
(2) Show that $\lambda \in \sigma_p(T)$ if and only if the range of $\overline{\lambda}I - T$ is not dense.
(3) Show that $\lambda \in \sigma_r(T) \Rightarrow \overline{\lambda} \in \sigma_p(T^*)$.
(4) Show that $\lambda \in \sigma_p(T) \Rightarrow \overline{\lambda} \in \sigma_r(T^*) \cup \sigma_p(T^*)$.

`Hint:` *(1) See Proposition XI.2(d). (2) Recall that $\ker T = (R(T^*))^{\perp}$.*

Problem 4. Let $H = \ell_2(\mathbb{N})$ (the complex version) and let $S : H \to H$ be defined by

$$S(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots).$$

(1) Compute S^*, S^*S and SS^* and show that S is a partial isometry.
(2) Show that $\sigma_p(S) = U(0,1)$ (the open unit disc) and deduce that $\sigma(S) = \sigma_{ap}(S) = \overline{U(0,1)}$.
(3) Deduce that $\sigma(S^*) = \overline{U(0,1)}$.
(4) Show that $\sigma_p(S^*) = \emptyset$.
(5) Deduce that $\sigma_r(S) = \emptyset$, $\sigma_r(S^*) = U(0,1)$ and $\sigma_c(S) = \sigma_c(S^*) = \{\lambda \in \mathbb{C}; |\lambda| = 1\}$.
(6) Show that for $\lambda \in U(0,1)$ we have $\|(\lambda I - S^*)x\| \geq (1 - |\lambda|) \|x\|$.
(7) Deduce that $\sigma_{ap}(S^*) = \{\lambda \in \mathbb{C}; |\lambda| = 1\}$.
(8) Given a complex unit λ, find a sequence (x_n) in H witnessing that $\lambda \in \sigma_{ap}(S^*)$.

`Hint:` *(3) Use Problem 3(1). (5) Use Problem 3(3,4). (8) Take the normalization of vectors* $(\lambda^n, \dots, \lambda, 1, 0, 0, \dots)$.

Problem 5. Let $H = \ell_2(\mathbb{Z})$ (the complex version) and let $S : H \to H$ be defined by

$$S((x_n)) = (x_{n-1}).$$

(1) Compute S^*, S^*S and SS^* and show that S is a unitary operator.
(2) Show that $\sigma_p(S) = \emptyset$.
(3) Show that $\sigma(S) = \sigma_{ap}(S) = \sigma_c(S) = \{\lambda \in \mathbb{C}; |\lambda| = 1\}$.
(4) Given a complex unit λ, find a sequence (x_n) in H witnessing that $\lambda \in \sigma_{ap}(S)$.
(5) Solve (2)–(4) for S^*.

`Hint:` *(3) Since* $S^* = S^{-1}$*, it follows from (2) that* $\sigma_p(S^*) = \emptyset$ *as well. Use Problem 3(3). (4) Modify the vectors from Problem 4(8).*

Problem 6. Let H be a (complex) Hilbert space and $T \in L(H)$. Show that $W(T^*) = \{\overline{\lambda}; \lambda \in W(T)\}$ and deduce that $w(T^*) = w(T)$.

Problem 7. (1) Let $H = \mathbb{R}^2$ be the two-dimensional real Hilbert space. Let $T \in L(H)$ be defined by $T(x_1, x_2) = (-x_2, x_1)$. Show that $W(T) = \{0\}$ and $w(T) = 0$.
(2) Let H be a real Hilbert space whose dimension is an even natural number. Find a unitary operator $T \in L(H)$ with $w(T) = 0$.
(3) Let H be an infinite dimensional real Hilbert space. Find a unitary operator $T \in L(H)$ with $w(T) = 0$.
(4) Let H be a real Hilbert space of a finite odd dimension. Let $T \in L(H)$ be an invertible operator. Show that $w(T) > 0$.

`Hint:` *(4) A square matrix of odd order with real entries has at least one real eigenvalue.*

Problem 8. Let $H = \mathbb{C}^2$ be the two-dimensional complex Hilbert space. Let $T \in L(H)$ be defined by $T(x_1, x_2) = (-x_2, x_1)$. Show that $W(T) = \{it; t \in [-1, 1]\}$ and deduce that $w(T) = 1$.

Problem 9. Let $H = \mathbb{C}^2$ be the two-dimensional complex Hilbert space. Let $T \in L(H)$ be defined by $T(x_1, x_2) = (0, x_1)$. Show that $W(T) = \overline{U(0, \frac{1}{2})}$. Deduce that $w(T) = \frac{1}{2}$ and hence the constant $\frac{1}{2}$ in Proposition XII.3(a) is optimal.

Problem 10. Let $H = \mathbb{C}^3$ be the three-dimensional complex Hilbert space. Let $T \in L(H)$ be defined by $T(x_1, x_2, x_3) = (0, x_1, x_2)$. Compute $W(T)$ and $w(T)$.

Hint: *Find maximum of the function* $a_1a_2 + a_2a_3$ *over* $\{(a_1, a_2, a_3) \in \mathbb{R}^3; a_1^2 + a_2^2 + a_3^2 = 1\}$ *(for example using Lagrange multipliers).*

Problem 11. (1) Let S be the operator from Problem 4. Show that $W(S) = U(0, 1)$.
(2) Let S be the operator from Problem 5. Show that $W(S) = U(0, 1)$.

Hint: *To prove '$\subset$' use the Cauchy-Schwarz inequality, including the criterion for equality. To prove '$\supset$' in (1) use the eigenvectors of S, in (2) use the same vectors completed by zeros.*

Problem 12. (1) Find a polarization formula (i.e., an analogue of Lemma XII.2) for self-adjoint operators on a real Hilbert space.
(2) Show that the formula is valid only for self-adjoint operators.

Problem 13. Let $H = \ell^2(\mathbb{N})$ (the complex version) and let (α_n) be a bounded sequence of complex numbers. For $\boldsymbol{x} = (x_n) \in H$ let $T(\boldsymbol{x}) = (\alpha_n x_n)$.

(1) Show that $T \in L(H)$, $\|T\| = \sup_n |\alpha_n|$, determine $\sigma(T)$ and $\sigma_p(T)$.
(2) Compute T^* and show that T is normal.
(3) Compute $|T|$ and find the polar decomposition of T.
(4) Let S be the operator from Problem 4. Let $T_1 = ST$ and $T_2 = S^*T$. Compute T_1^* and T_2^*. Are these operators normal?
(5) Compute $|T_1|$, $|T_1^*|$, $|T_2|$, $|T_2^*|$.
(6) Compute the polar decompositions of operators T_1, T_1^*, T_2, T_2^*.

Problems to Section XII.2 – unbounded operators

Problem 14. Let T be a densely defined closed operator from a Banach space X to a Banach space Y. Show that T is everywhere defined if and only if T is continuous.

Problem 15. Let $X = \ell^p$ where $p \in [1, \infty)$ or $X = c_0$. Let $\boldsymbol{z} = (z_n)$ be a sequence of (real or complex) numbers. Let

$$D(M_{\boldsymbol{z}}) = \{(x_n) \in X; (x_n z_n) \in X\}$$

and define the operator $\mathcal{M}_{\boldsymbol{z}}$ by

$$M_{\boldsymbol{z}}((x_n)) = (x_n z_n), \quad (x_n) \in D(M_{\boldsymbol{z}}).$$

(1) Show that $M_{\boldsymbol{z}}$ is a densely defined closed operator on X.
(2) Show that $M_{\boldsymbol{z}}$ is bounded (hence everywhere defined) if and only if the sequence $\boldsymbol{z}$ is bounded. Show that in this case $\|M_{\boldsymbol{z}}\| = \|\boldsymbol{z}\|_\infty$.

Problem 16. Let $X = \ell^\infty$ and define the operator $M_{\boldsymbol{z}}$ in the same way as in Problem 15.

(1) Show that $M_{\boldsymbol{z}}$ is bounded if and only if the sequence $\boldsymbol{z}$ is bounded. Show that in this case $M_{\boldsymbol{z}}$ is everywhere defined and $\|M_{\boldsymbol{z}}\| = \|\boldsymbol{z}\|_\infty$.
(2) Show that $M_{\boldsymbol{z}}$ is a closed operator.
(3) Show that $M_{\boldsymbol{z}}$ is not densely defined unless $\boldsymbol{z}$ it bounded.

Problem 17. Let (Ω, Σ, μ) be a measure space with μ semifinite (i.e., whenever $A \in \Sigma$ is such that $\mu(A) > 0$, then there is $B \in \Sigma$ such that $B \subset A$ and $0 < \mu(B) < \infty$). Let $X = L^p(\mu)$ where $p \in [1, \infty)$. Let g be a measurable function on Ω. Set

$$D(M_g) = \{f \in X; fg \in X\}$$

and define the operator M_g by

$$M_g(f) = fg, \quad f \in D(M_g).$$

(1) Show that M_g is a densely defined closed operator on X.
(2) Show that M_g is bounded (hence everywhere defined) if and only if the function g is essentially bounded. Show that in this case $\|M_g\| = \|g\|_\infty$.

`Hint:` *(1) Use (among others) the density of simple integrable functions in $L^p(\mu)$. (2) Use suitable characteristic functions.*

Problem 18. Let $X = L^\infty((0,1))$ and define the operator M_g in the same way as in Problem 17.

(1) Show that M_g is bounded if and only if the function g is essentially bounded. Show that in this case M_g is everywhere defined and $\|M_g\| = \|g\|_\infty$.
(2) Show that M_g is a closed operator.
(3) Show that M_g is not densely defined unless g it essentially bounded.

Problem 19. Let $X = \ell^p$, where $p \in (1, \infty)$. Let

$$Y = \{(x_n) \in X; (nx_n) \in X \;\&\; \sum_{n=1}^{\infty} x_n = 0\}.$$

(1) Show that $(nx_n) \in X$ implies $(x_n) \in \ell^1$ and deduce that Y is a well-defined linear subspace of X.
(2) Show that Y is dense in X.
(3) Define the operator T by $T((x_n)) = (nx_n)$ for $(x_n) \in D(T) = Y$. Show that T is a closed operator.

`Hint:` *(1) Use the Hölder inequality. (2) Approximate any finitely supported vector by an element of Y. (3) Use the definitions and the Hölder inequality.*

Problem 20. Let $X = L^p((1, \infty))$, where $p \in (1, \infty)$. Let $\varphi(t) = t$ for $t \in (1, \infty)$ and

$$Y = \{f \in X; \varphi \cdot f \in X \;\&\; \int_1^\infty f = 0\}.$$

(1) Show that $\varphi \cdot f \in X$ implies $f \in L^1((1, \infty))$ and deduce that Y is a well-defined linear subspace of X.
(2) Show that Y is dense in X.
(3) Define the operator T by $T(f) = \varphi \cdot f$ for $(x_n) \in D(T) = Y$. Show that T is a closed operator.

`Hint:` *(1) Use the Hölder inequality. (2) Approximate characteristic functions of bounded measurable sets by elements of Y and use density of simple integrable functions in X. (3) Use the definitions and the Hölder inequality.*

Problem 21. Let $X = L^p((1,\infty))$, where $p \in (1,\infty)$. Let $\varphi(t) = [t]$ (the integer part of t) for $t \in (1,\infty)$ and

$$Y = \{f \in X; \varphi \cdot f \in X \ \& \ \int_1^\infty f = 0\}.$$

(1) Show that $\varphi \cdot f \in X$ implies $f \in L^1((1,\infty))$ and deduce that Y is a well-defined linear subspace of X.
(2) Show that Y is dense in X.
(3) Define the operator T by $T(f) = \varphi \cdot f$ for $(x_n) \in D(T) = Y$. Show that T is a closed operator.

`Hint:` *Proceed similarly as in Problem 20.*

Problem 22. Let $X = L^p((0,1))$ where $p \in [1,\infty)$. Set

$$Y = \{f \in AC([0,1]); f' \in X\},$$

where by $AC([0,1])$ we denote the space of functions which are absolutely continuous on $[0,1]$. Define operators T_j, $j = 1,\dots,6$, all of them by the same formula $T_j(f) = f'$, with domains

$$\begin{aligned} D(T_1) &= Y, & D(T_4) &= \{f \in Y; f(0) = f(1) = 0\}, \\ D(T_2) &= \{f \in Y; f(0) = 0\}, & D(T_5) &= \{f \in Y; f(0) = f(1)\}, \\ D(T_3) &= \{f \in Y; f(1) = 0\}, & D(T_6) &= \{f \in Y; f(0) = -f(1)\}. \end{aligned}$$

Show that all these operators are densely defined and closed (consider Y as a subspace of X).

`Hint:` *To prove they are densely defined use the density of test functions in $L^p((0,1))$. To prove they are closed use the boundedness of the operator $V : X \to X$ defined by $V(f)(t) = \int_0^t f$, $t \in (0,1)$, $f \in X$.*

Problem 23. Let $X = L^p((0,\infty))$ where $p \in [1,\infty)$. Set

$$Y = \{f \in AC_{loc}([0,\infty)); f \in X \ \& \ f' \in X\},$$

where by $AC_{loc}([0,\infty))$ we denote the space of functions defined on $[0,\infty)$ which are absolutely continuous on $[0,R]$ for each $R \in (0,\infty)$ (considered as a subspace of X). Define operators T_j, $j = 1,2$, both of them by the same formula $T_j(f) = f'$, with domains

$$D(T_1) = Y, \qquad D(T_2) = \{f \in Y; f(0) = 0\}.$$

(1) Show that $\lim_{t\to\infty} f(t) = 0$ for each $f \in Y$.
(2) Show that T_1 and T_2 are densely defined.
(3) Show that T_1 and T_2 are closed.

`Hint:` *(1) Let $f \in Y$. Consider the function $g = |f|^p$. Compute g' and show that $g' \in L^1((0,\infty))$ (using the Hölder inequality). Deduce that g has a finite limit in $+\infty$ and that this limit has to be zero as $g \in L^1((0,\infty))$. (2) Use the density of test functions in $L^p((0,\infty))$. (3) Use the analogue of the operator V from the hint to Problem 22 on intervals $(0,R)$ for $R \in (0,\infty)$.*

Problem 24. Let $X = L^p(\mathbb{R})$ where $p \in [1, \infty)$. Set

$$Y = \{f \in AC_{loc}(\mathbb{R}); f \in X \ \&\ f' \in X\},$$

where by $AC_{loc}(\mathbb{R})$ we denote the space of functions defined on $\mathbb{R}$ which are absolutely continuous on $[-R, R]$ for each $R \in (0, \infty)$ (considered as a subspace of X). Fix a closed set $F \subset \mathbb{R}$ of Lebesgue measure zero. Define operators T_j, $j = 1, 2, 3$, all of them by the same formula $T_j(f) = f'$, with domains

$$D(T_1) = Y, \qquad D(T_2) = \{f \in Y; f(0) = 0\}, \qquad D(T_3) = \{f \in Y; f|_F = 0\}.$$

(1) Show that $\lim_{t\to\pm\infty} f(t) = 0$ for each $f \in Y$.
(2) Show that T_1, T_2 and T_3 are densely defined.
(3) Show that T_1, T_2 and T_3 are closed.

`Hint:` *(1) Use Problem 23. (2) Using density of test functions in $L^p(J)$ for any open interval J and the assumption that F has measure zero show that the test functions with support disjoint with F are dense in X. (3) Use an analogous approach to that in Problem 23.*

Problems to Section XII.3 – spectrum of an unbounded operators

Problem 25. Consider the operator $M_{\boldsymbol{z}}$ from Problem 15.

(1) Show that $\sigma(M_{\boldsymbol{z}}) = \overline{\{z_n; n \in \mathbb{N}\}}$.
(2) Show that the eigenvalues of $M_{\boldsymbol{z}}$ are exactly the values z_n, $n \in \mathbb{N}$.
(3) Show that $\sigma(M_{\boldsymbol{z}})$ can be any nonempty closed subset of $\mathbb{C}$.

Problem 26. Consider the operator M_g from Problem 17.

(1) Show that $\sigma(M_g)$ is the essential range of g.
(2) Show that $\lambda \in \mathbb{C}$ is an eigenvalue of M_g if and only if $\mu(g^{-1}(\lambda)) > 0$.
(3) Show that, in case (Ω, Σ, μ) is the interval $[0, 1]$ with the Lebesgue measure, $\sigma(M_g)$ can be any nonempty closed subset of $\mathbb{C}$.

Problem 27. Let S, T be two closed operators on a Banach space X such that $S \subsetneqq T$.

(1) Show that any eigenvalue of S is also an eigenvalue of T, but the converse need not hold.
(2) Show that $\rho(S) \cap \rho(T) = \emptyset$, in other words $\sigma(S) \cup \sigma(T) = \mathbb{C}$.

`Hint:` *(1) To find a counterexample look at Problem 28.*

Problem 28. Compute the eigenvalues and the spectrum of the following operators:

(1) The operator T from Problem 19.
(2) The operator T on $\ell^p(\mathbb{Z})$ (where $p \in [1, \infty)$) defined by

$$T((x_n)_{n\in\mathbb{Z}}) = (|n| \cdot x_n)_{n\in\mathbb{Z}}, \quad (x_n) \in D(T) = \{(y_n) \in \ell^p(\mathbb{Z}); (|n|\, y_n) \in \ell^p(\mathbb{Z}) \ \&\ \sum_{n\in\mathbb{Z}} y_n = 0\}.$$

(3) The operator T from Problem 20.
(4) The operator T from Problem 21.

Problem 29. Compute the eigenvalues and the spectrum of the operators from Problem 22.

`Hint:` *Use the standard methods of solving differential equations.*

Problems to Section XII.4 - operators on a Hilbert space

Problem 30. Let T be a closed densely defined operator. Show that

$$\sigma(T^*) = \{\overline{\lambda}; \lambda \in \sigma(T)\}.$$

`Hint:` *Use Lemma XII.20.*

Problem 31. Consider the operator $M_{\boldsymbol{z}}$ from Problem 15 on ℓ^2. Show that $(M_{\boldsymbol{z}})^* = M_{\overline{\boldsymbol{z}}}$ and characterize sequences $\boldsymbol{z}$ for which $M_{\boldsymbol{z}}$ is selfadjoint.

Problem 32. Consider the operator M_g from Problem 17 on $L^2(\mu)$. Show that $(M_g)^* = M_{\overline{g}}$ and characterize functions g for which M_g is selfadjoint.

Problem 33. Consider the operators T_j, $j = 1, \dots, 6$, from Problem 22 on $L^2((0,1))$.

(1) Show that $T_1^* = -T_4$, $T_2^* = -T_3$, $T_3^* = -T_2$, $T_4^* = -T_1$, $T_5^* = -T_5$, $T_6^* = -T_6$.
(2) Deduce that iT_5 and iT_6 are selfadjoint and that iT_4 is symmetric.

`Hint:` *(1) To prove the inclusions '$\supset$' use integration by parts. To prove '$\subset$' proceed as follows: Let $g \in D(T_j^*)$. Then there is $h \in L^2((0,1))$ such that $\langle T_j f, g\rangle = \langle f, h\rangle$ for any $f \in D(T_j)$. Set $H(t) = \int_0^t h$, $t \in [0,1]$. Apply integration by parts. Note that $\mathscr{D}((0,1)) \subset D(T_j)$ and deduce that the distributive derivative of $g + H$ on $(0,1)$ is zero, thus $g + H$ is almost everywhere equal to a constant. So, $g \in AC([0,1])$ and $H = g(0) - g$. Plug this to the computation and conclude.*

Problem 34. Let T be a symmetric operator on a Hilbert space which is not maximal symmetric (i.e., it admits a proper symmetric extension). Show that $\sigma(T) = \mathbb{C}$.

`Hint:` *Combine the ideas from Problem 27 and Lemma XII.24.*

Problem 35. Consider the operators T_1 and T_2, from Problem 23 on $L^2((0,\infty))$.

(1) Show that $T_1^* = -T_2$ and $T_2^* = -T_1$.
(2) Deduce that iT_2 is symmetric.
(3) Compute the eigenvalues and the spectrum of T_1 and T_2.
(4) Deduce that iT_2 is a maximal symmetric operator.

`Hint:` *(1) Proceed similarly as in Problem 33, use integration by parts on $[0,r]$, take the limit for $r \to \infty$ and use Problem 23(1). (3) To compute the eigenvalues use the standard methods of solving differential equations. To determine when $\lambda I - T_2$ is onto use first Lemma XII.24 to show that, in case $\operatorname{Re}\lambda \neq 0$, the surjectivity is equivalent to the density of range. To determine when the range is dense use Proposition XII.18 and the knowledge of eigenvalues of T_1. Finally, to describe $\sigma(T_1)$ use Problem 30. (4) Combine the result of (3) with Problem 34.*

Problem 36. Consider the operators T_1, T_2 and T_3 from Problem 24 on $L^2(\mathbb{R})$.

(1) Show that $T_1^* = -T_1$, deduce that iT_1 is selfadjoint and T_2, T_3 are symmetric.
(2) Show that

$$D(T_2^*) = \{f \in L^2(\mathbb{R}); \forall R > 0 : f \text{ is absolutely continuous both on } (0,R) \text{ and on } (-R,0) \ \&\ f' \in L^2(\mathbb{R})\}$$

and that $T_2^*(f) = -f'$ for $f \in D(T_2^*)$. (The derivative is taken in the sense "almost everywhere", not in the sense of distributions.)
(3) Show that

$$D(T_3^*) = \{f \in L^2(\mathbb{R}); f \text{ is absolutely continuous on each bounded open interval disjoint with F} \ \&\ f' \in L^2(\mathbb{R})\}$$

and that $T_3^*(f) = -f'$ for $f \in D(T_3^*)$. (The derivative is taken in the sense "almost everywhere", not in the sense of distributions.)

(4) Compute the eigenvalues and the spectrum of T_1.

(5) Determine the eigenvalues and the spectrum of T_2 and T_3.

`Hint:` *(1) Proceed similarly as in Problem 35. (2) Proceed similarly as in Problem 35 separately on $(0, \infty)$ and on $(-\infty, 0)$. (3) $\mathbb{R} \setminus \mathbb{C}$ is an open set, so it is a countable disjoint union of open intervals. Proceed on each of these intervals separately - on bounded ones similarly as in Problem 33, on unbounded ones similarly as in Problem 35. (4) Use (1), Theorem XII.25 and standard methods of solving differential equations. (5) Use (1), (4) and Problem 27.*

Problem 37. Consider the operator T from Problem 19 on ℓ^2.

(1) Define

$$\psi((x_n)) = \lim_{n\to\infty} nx_n, \qquad (x_n) \in D(\psi) = \{(y_n) \in \ell^2; \lim_{n\to\infty} ny_n \text{ exists and is finite}\}.$$

Show that ψ is a densely defined linear functional on ℓ^2.

(2) Show that

$$D(T^*) = \{(x_n) \in D(\psi); (x_k - \psi((x_n)))_{k=1}^{\infty} \in \ell_2\}$$

and that

$$T^*((x_n)) = (x_k - \psi((x_n)))_{k=1}^{\infty}, \quad (x_n) \in D(T^*).$$

`Hint:` *(2) The inclusion '$\supset$' can be proved by an easy computation starting from definitions. To show '$\subset$' proceed as follows: Let $\boldsymbol{y} \in D(T^*)$. Then there exists $\boldsymbol{z} \in \ell^2$ such that $\langle T\boldsymbol{x}, \boldsymbol{y}\rangle = \langle \boldsymbol{x}, \boldsymbol{z}\rangle$ for $\boldsymbol{x} \in D(T)$. Apply for $\boldsymbol{x} = \boldsymbol{e}_1 - \boldsymbol{e}_n$, $n \geq 2$, and deduce that $z_n = z_1 - y_1 + ny_n$, $n \geq 2$. Using the assumption that $\boldsymbol{z} \in \ell_2$ and hence $z_n \to 0$ compute $\lim ny_n$ and conclude.*

Problem 38. Compute T^* for the operator T from Problem 28(2) on $\ell^2(\mathbb{Z})$.

`Hint:` *Proceed similarly as in Problem 37.*

Problem 39. Consider the operator T from Problem 20 on $L^2((1, \infty))$.

(1) Set

$$D(\psi) = \{f \in L^2((1, \infty)); \exists C \in \mathbb{C} : C + \varphi \cdot f \in L^2((1, \infty))\}.$$

Show that $D(\psi)$ is a dense linear subspace of $L^2((1, \infty))$, that for $f \in D(\psi)$ the constant C from the definition is uniquely determined and the mapping

$$\psi : f \mapsto \text{ the respective } C$$

is a linear functional defined on $D(\psi)$.

(2) Show that $D(T^*) = D(\psi)$ and $T^*(f) = \psi(f) + \varphi \cdot f$ for $f \in D(T^*)$.

`Hint:` *The inclusion '$\supset$' can be proved by an easy computation starting from definitions. To show '$\subset$' proceed as follows: Let $g \in D(T^*)$. Then there exists $h \in L^2((1, \infty))$ such that $\langle Tf, g\rangle = \langle f, h\rangle$ for $f \in D(T)$. Apply for $f = (r-1)\chi_{(1,2)} - \chi_{(1,r)}$. Differentiate the resulting equality with respect to r (use the known fact on differentiating the indefinite integral) and deduce that $h(r) = \int_1^2 (h - fg) + \varphi(r) \cdot g(r)$ almost everywhere. Then complete the argument.*

Problem 40. Compute T^* for the operator T from Problem 21 on $L^2((1, \infty))$.

`Hint:` *Proceed similarly as in Problem 39.*

Problem 41. Set $\varphi(t) = t$ for $t \in \mathbb{R}$. Compute the adjoint of the operator T on $L^2(\mathbb{R})$ defined by

$$T(f) = \varphi \cdot f, \quad f \in D(T) = \{g \in L^2(\mathbb{R}); \varphi \cdot g \in L^2(\mathbb{R}) \ \& \ \int_{-\infty}^{\infty} f(t)e^{it} = 0\}.$$

`Hint:` *Proceed similarly as in Problem 39.*

Problems to Section XII.5 - symmetric operators and Cayley transform

Problem 42. Let S be a self-adjoint operator on a Hilbert space and let C_S be its Cayley transform.

(1) Show that S is bounded (i.e., everywhere defined) if and only if $1 \notin \sigma(C_S)$.
(2) Suppose that S is bounded. Describe C_S using continuous functional calculus applied to S. And, conversely, describe S using continuous functional calculus applied to C_S.
(3) Suppose S is bounded. Using the descriptions in (2) describe the relationship of $\sigma(S)$ and $\sigma(C_S)$.
(4) Show that the relationship found in (3) is valid also in case S is unbounded.

`Hint:` *(4) For $\lambda \neq 1$ show by a direct computation that $\lambda I - C_S$ is invertible if and only if $(\lambda - 1)S + i(\lambda + 1)I$ is one-to-one and surjective.*

Problem 43. Let S be a closed densely defined symmetric operator on a Hilbert space. Show that there are the following posibilites for the spectrum of S.

(a) $\sigma(S) \subset \mathbb{R}$ if S is selfadjoint.
(b) $\sigma(S) = \mathbb{C}$ if S is not maximal.
(c) $\sigma(S) = \{\lambda \in \mathbb{C}; \operatorname{Im} \lambda \geq 0\}$ or $\sigma(S) = \{\lambda \in \mathbb{C}; \operatorname{Im} \lambda \leq 0\}$ if S is maximal symmetric but not selfadjoint.

`Hint:` *To show (a) and (b) use Theorem XII.25 and Problem 34. Assume that S is maximal but not self-adjoint. Using Corollary XII.26 and remark (2) at the end of Section XII.5 to show that $\sigma(S)$ contains exactly one of the numbers $\pm i$. Applying this observation to the operator $\frac{1}{\beta}(\alpha I - S)$ deduce that $\sigma(S)$ contains exactly one of the numbers $\alpha \pm \beta i$ whenever $\alpha \in \mathbb{R}$ and $\beta > 0$. Conclude using closedness of $\sigma(S)$ and connectedness of halfplanes.*

Problem 44. Consider the operators M_g from Problem 17 on $L^2(\mu)$.

(1) Suppose M_g is self-adjoint. Show that its Cayley transform is again of the form M_h and determine h.
(2) Characterize functions g for which M_g is a unitary operator such that $I - M_g$ is one-to-one.
(3) Let g be as in (2). Find h such that M_g is the Cayley transform of M_h.

Problem 45. Compute the Cayley transforms of the following operators:

(1) The operators iT_5 and iT_6, where T_5 and T_6 are the operators from Problem 22 on $L^2((0,1))$ (by Problem 33 they are selfadjoint).
(2) The operator iT_4, where T_4 is the operator from Problem 22 on $L^2((0,1))$ (by Problem 33 it is symmetric).
(3) The operator iT_2, where T_2 is the operator from Problem 23 on $L^2((0,\infty))$ (by Problem 35 it is symmetric).
(4) The operator iT_1, where T_1 is the operator from Problem 24 on $L^2(\mathbb{R})$ (by Problem 36 it is selfadjoint).
(5) The operator iT_2 and iT_3, where T_2 and T_3 are the operators from Problem 24 on $L^2(\mathbb{R})$ (by Problem 36 they are symmetric).

`Hint:` *The solution has two parts – to determine the domain of the Cayley transform and to compute the formula. The formula can be computed in all the cases using standard methods of solving differential equations and differentianting indefinite integrals. The domain in cases (1,4) is the whole space by Theorem XII.30. In the remaing cases the domain can be described by* $D(C_S) = R(S + iI) = \operatorname{Ker}(S^* - iI)^\perp$ *(use Theorem XII.27(a,c) and Proposition XII.18). To describe it more concretely, use the knowledge of the spectrum and eigenvalues of the respective operators. The range of the Cayley transform can be determined similarly.*

Problem 46. Compute the Cayley transforms of the following operators:

(1) The operator T from Problem 19 on ℓ^2.
(2) The operator T from Problem 20 on $L^2((1,\infty))$.
(3) The operator T from Problem 21 on $L^2((1,\infty))$.

`Hint:` *The formula for the Cayley transform may de deduced using Problem 44. The domains can be found using the knowledge of spectra similarly as in Problem 45.*