III.1 Power series of several complex variables

Notation:

- Let $\boldsymbol{x} \in \mathbb{C}^n$, $\boldsymbol{y} \in \mathbb{C}^n$ and $c \in \mathbb{C}$. Then we set
  - $\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \dots, x_n + y_n)$,
  - $\boldsymbol{x} \cdot \boldsymbol{y} = (x_1 y_1, \dots, x_n y_n)$,
  - $c\boldsymbol{x} = (cx_1, \dots, cx_n)$.
- Let $\boldsymbol{x} \in \mathbb{C}^n$ and $\alpha \in \mathbb{N}_0^n$. Then we set $\boldsymbol{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where we use the convention that $0^0 = 1$.
- For $\alpha \in \mathbb{N}_0^n$ we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Definition. By a **power series of n variables centered at 0** we mean a series of the form

$$\sum_{\alpha \in \mathbb{N}_0^n} c_\alpha \boldsymbol{x}^\alpha,$$

where $c_\alpha \in \mathbb{C}$ for each $\alpha \in \mathbb{N}_0^n$.

Remark: On the index set $\mathbb{N}_0^n$ there is no canonical order. Therefore by a convergence of the above power series we mean the absolute convergence. I.e., the power series given above converges at a point $\boldsymbol{x}$ if and only if

$$\sup\{\sum_{\alpha \in F} |c_\alpha \boldsymbol{x}^\alpha| : F \subset \mathbb{N}_0^n \text{ finite}\} < +\infty.$$

The sum of this series is then the limit of partial sums for any ordering of the elements of the series or, equivalently, the limit of the net

$$\sum_{\alpha \in F} c_\alpha \boldsymbol{x}^\alpha, \qquad F \subset \mathbb{N}_0^n \text{ finite},$$

where finite sets are ordered by inclusion.

Proposition 1.

(1) *The series $\sum_{\alpha \in \mathbb{N}_0^n} \boldsymbol{x}^\alpha$ converges if and only if $|x_j| < 1$ for each $j \in \{1, \dots, n\}$.*

(2) *Let us consider the series $\sum_{\alpha \in \mathbb{N}_0^n} c_\alpha \boldsymbol{x}^\alpha$. Let $\boldsymbol{x} \in \mathbb{C}^n$ be a point with non-zero coordinates such that*

$$\sup_{\alpha \in \mathbb{N}_0^n} |c_\alpha \boldsymbol{x}^\alpha| < +\infty.$$

Then the series converges locally uniformly on the set

$$\{\boldsymbol{y} \in \mathbb{C}^n : |y_j| < |x_j| \text{ for } j = 1, \dots, n\}.$$

Definition. Let $A \subset \mathbb{C}^n$. The set A is said to be

- a **Reinhardt set** if $\boldsymbol{y} \cdot \boldsymbol{x} \in A$ whenever $\boldsymbol{x} \in A$ and $\boldsymbol{y} \in \mathbb{T}^n$,
- a **complete Reinhardt set** if $\boldsymbol{y} \cdot \boldsymbol{x} \in A$ whenever $\boldsymbol{x} \in A$ and $\boldsymbol{y} \in \overline{\mathbb{D}}^n$.

A Reinhardt set A is said to be **logarithmically convex** if

$$\log A = \{(\log|x_1|, \dots, \log|x_n|) : \boldsymbol{x} \in A \cap (\mathbb{C} \setminus \{0\})^n\}$$

is a convex subset of $\mathbb{R}^n$.

Proposition 2. *Let $A \subset \mathbb{C}^n$ be a complete Reinhardt set containing at least one point with non-zero coordinates. Then*

$$0 \in \operatorname{Int} A, \qquad \operatorname{Int} A = \operatorname{Int} \overline{A}.$$

Further, if $\boldsymbol{x} \in \overline{A}$ has non-zero coordinates, then $x \in \overline{\operatorname{Int} A}$.

Theorem 3. *Let S be a power series of the above form. Consider the following sets*

$$\begin{aligned}
\mathcal{B}_S &= \{\boldsymbol{x} \in \mathbb{C} : \sup_{\alpha \in \mathbb{N}_0^n} |c_\alpha x^\alpha| < +\infty\} \\
\mathcal{C}_S &= \{\boldsymbol{x} \in \mathbb{C} : \sum_{\alpha \in \mathbb{N}_0^n} |c_\alpha x^\alpha| < +\infty\} \\
\mathcal{D}_S &= \operatorname{Int} \mathcal{B}_S.
\end{aligned}$$

Then

$$\mathcal{D}_S \subset \mathcal{C}_S \subset \mathcal{B}_S.$$

All these sets are complete Reinhardt sets, the sets $\mathcal{B}_S$ and $\mathcal{D}_S$ are moreover logarithmically convex. The series S converges locally uniformly on $\mathcal{D}_S$.

Definition. The set $\mathcal{D}_S$ is called the **domain of convergence** of the series S.

Theorem 4. *Let $\Omega \subset \mathbb{C}^n$ be a logarihtmically convex complete Reinhardt domain. Then Ω is the domain of convergence of a power series.*

Theorem 5. *Let us consider the following two power series*

$$S = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha \boldsymbol{x}^\alpha \qquad a \qquad S' = \sum_{\alpha \in \mathbb{N}_0^n, \alpha_j \geq 1} \alpha_j c_\alpha \boldsymbol{x}^{\alpha - \boldsymbol{e}^j}.$$

Then $\mathcal{D}_S \subset \mathcal{D}_{S'}$. By $\boldsymbol{e}^j$ we denote the j-th canonical vector, i.e.,

$$\boldsymbol{e}^j = (0, \dots, 0, \underset{j\text{-th position}}{1}, 0, \dots, 0).$$