6.2. **Antiderivatives.**

Definition. Let a function f be defined on an open interval I. We say that a function F is an *antiderivative of f on I*, if for each $x \in I$ there exists $F'(x)$ and $F'(x) = f(x)$.

Theorem 6.7. *Let F and G be antiderivatives of f on an open interval I. Then there exists $c \in \mathbb{R}$ such that $F(x) = G(x) + c$ for each $x \in I$.*

Notation: The set of all antiderivatives of f on an open interval I is denoted by $\int f(x)\,\mathrm{d}x$. The fact that F is an antiderivative of f on I is expressed by

$$\int f(x)\,\mathrm{d}x \stackrel{c}{=} F(x), \qquad x \in I.$$

Proposition 6.8 (table of basic antiderivatives)**.**

- $\int x^n\,\mathrm{d}x \stackrel{c}{=} \frac{x^{n+1}}{n+1}$ *on* $\mathbf{R}$ *(for* $n \in \mathbf{N} \cup \{0\}$*);*
- $\int x^\alpha\,\mathrm{d}x \stackrel{c}{=} \frac{x^{\alpha+1}}{\alpha+1}$ *on* $(0,+\infty)$ *(for* $\alpha \in \mathbf{R} \setminus \{-1\}$*, for* $\alpha \in \mathbf{Z}$*,* $\alpha < -1$ *also on* $(-\infty,0)$*);*
- $\int \frac{1}{x}\,\mathrm{d}x \stackrel{c}{=} \log x$ *on* $(0,+\infty)$*,* $\int \frac{1}{x}\,\mathrm{d}x = \log(-x)$ *on* $(-\infty,0)$*;*
- $\int \exp x\,\mathrm{d}x \stackrel{c}{=} \exp x$ *on* $\mathbf{R}$*;*
- $\int \sin x\,\mathrm{d}x \stackrel{c}{=} -\cos x$ *on* $\mathbf{R}$*,* $\int \cos x\,\mathrm{d}x \stackrel{c}{=} \sin x$ *on* $\mathbf{R}$*;*
- $\int \frac{1}{\cos^2 x}\,\mathrm{d}x \stackrel{c}{=} \operatorname{tg} x$ *on any of the intervals* $(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi)$*,* $k \in \mathbf{Z}$*;*
- $\int \frac{1}{\sin^2 x}\,\mathrm{d}x \stackrel{c}{=} -\operatorname{cotg} x$ *on any of the intervals* $(k\pi,(k+1)\pi)$*,* $k \in \mathbf{Z}$*;*
- $\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \stackrel{c}{=} \arcsin x$ *on* $(-1,1)$*;*
- $\int \frac{1}{1+x^2}\,\mathrm{d}x \stackrel{c}{=} \operatorname{arctg} x$ *on* $\mathbf{R}$*.*

Theorem 6.9 (existence of antiderivative)**.** *Let f be a continuous function on an open interval I. Then f admits an antiderivative on I.*

Theorem 6.10. *Let f have on an open interval I an antiderivative F, let a function g have on I an antiderivative G, and let $\alpha, \beta \in \mathbf{R}$. Then the function $\alpha F + \beta G$ is an antiderivative of $\alpha f + \beta g$ on I.*

Theorem 6.11 (substitution)**.** *(i) Let F be an antiderivative of f on (a,b). Let φ be a function defined on an interval (α,β) with values in (a,b) and such that φ has at each point $t \in (\alpha,\beta)$ finite derivative. Then we have*

$$\int f(\varphi(t))\varphi'(t)\,dt \stackrel{c}{=} F(\varphi(t)) \text{ on } (\alpha,\beta).$$

(ii) Let φ be a strictly monotone function on an interval (α,β) which has at each point of (α,β) finite derivative and satisfies $\varphi((\alpha,\beta)) = (a,b)$. Let f be a function continuous on (a,b). If we have

$$\int f(\varphi(t))\varphi'(t)\,dt \stackrel{c}{=} G(t) \text{ on } (\alpha,\beta),$$

then

$$\int f(x)\,dx \stackrel{c}{=} G(\varphi^{-1}(x)) \text{ on } (a,b).$$

Theorem 6.12 (integration by parts)**.** *Let I be an open interval and let functions f and g be continuous on I. Let F be an antiderivative of f on I and G be an antiderivative of g on I. Then we have*

$$\int g(x)F(x)\,dx = G(x)F(x) - \int G(x)f(x)\,dx \text{ na } I.$$