6.4. **Generalized Riemann integral.**

Remark. In this section the Riemann integral of f over $\langle a, b\rangle$ will be denoted by $(R)\int_a^b f(x)\,\mathrm{d}x$.

Definition. Let f be a function defined on an open interval (a, b), where $a \in \mathbf{R} \cup \{-\infty\}$ and $b \in \mathbf{R} \cup \{+\infty\}$. Let f be Riemann-integrable over each closed subinterval of (a, b). Let $c \in (a, b)$ be fixed. The *generalized Riemann integral over* (a, b) is defined by the formula

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{y\to a+} (R)\int_y^c f(x)\,\mathrm{d}x + \lim_{y\to b-} (R)\int_c^y f(x)\,\mathrm{d}x,$$

provided the two limits exist and their sum is defined.

Remark.

(1) The existence and the value of generalized Riemann integral do not depend on the particular choice of c.
(2) If f is Riemann-integrable over $\langle a, b\rangle$, then the generalized Riemann integral of f over (a, b) eixsts and is equal to $(R)\int_a^b f(x)\,\mathrm{d}x$.
(3) The generalized Riemann integral can assume also the values $-\infty$ or $+\infty$.

Theorem 6.14. *Let f be a function continuous on an open interval (a, b). Let F be an antiderivative of f on (a, b). Then*

$$\int_a^b f(x)\,\mathrm{d}x = (\lim_{y\to b-} F(y)) - (\lim_{y\to a+} F(y)),$$

whenever either left-hand side or the right-hand side is defined.

Remark. The expression on the right-hand side is denoted by $[F]_a^b$ and called *generalized increment of F over (a, b).*

Theorem 6.15 (integration by parts for definite integral)**.** *Left f and g be functions defined on (a, b), which have continuous derivative on (a, b). Then*

$$\int_a^b f'(x)g(x)\,\mathrm{d}x = [fg]_a^b - \int_a^b f(x)g'(x)\,\mathrm{d}x,$$

provided the right-hand side is defined.

Theorem 6.16 (substitution for definite integral)**.** *Let f be a function continuous on an interval (a, b), φ be a function defined on (α, β). Suppose that φ is strictly monotone on (α, β), has continuous derivative on (α, β) and maps (α, β) onto (a, b). Then*

$$\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f(\varphi(t)) \cdot |\varphi'(t)|\,\mathrm{d}t,$$

whenever at least one of these integrals is defined.