7. Infinite series

7.1. Basic notions.

Definition. Let $\{a_n\}$ be a sequence of real numbers.

- Symbol $\sum_{n=1}^{\infty} a_n$ is called an *infinite series*.
- The element a_n is called n*-th member* of the series $\sum_{n=1}^{\infty} a_n$.
- For $m \in \mathbf{N}$ we set
$$s_m = a_1 + a_2 + \cdots + a_m.$$
The number s_m is called m*-th partial sum* of the series $\sum_{n=1}^{\infty} a_n$.
- The *sum* of infinite series $\sum_{n=1}^{\infty} a_n$ is defined as the limit of the sequence $\{s_m\}$, if such a limit exists.
- The sum of the series is denoted by the symbol $\sum_{n=1}^{\infty} a_n$.
- We say that a series *converges*, if its sum is a real number. In the opposite case, we say that the series *diverges*.

Theorem 7.1 (necessary condition)**.** *If a series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim a_n = 0$.*

Remark. Suppose that $\alpha \in \mathbf{R}$ and a series $\sum_{n=1}^{\infty} a_n$ converges. Then the series $\sum_{n=1}^{\infty} \alpha a_n$ converges and it holds $\sum_{n=1}^{\infty} \alpha a_n = \alpha \sum_{n=1}^{\infty} a_n$. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then the series $\sum_{n=1}^{\infty}(a_n + b_n)$ converges and it holds $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

7.2. Series with nonnegative members and absolute convergence.

Remark. Let $\sum_{n=1}^{\infty} a_n$ be a series with nonnegative members (i.e., $a_n \geq 0$ for each $n \in \mathbf{N}$). Then this series has a sum – either it converges or it has sum $+\infty$.

Theorem 7.2. *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series satisfying $0 \leq a_n \leq b_n$ for each $n \in \mathbf{N}$.*

(i) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Theorem 7.3. *Let $\{a_n\}$ be a sequence of real numbers. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

Definition. We say that $\sum_{n=1}^{\infty} a_n$ is *absolute convergent*, if $\sum_{n=1}^{\infty} |a_n|$ converges. If $\sum_{n=1}^{\infty} a_n$ converges but not absolutely, then $\sum_{n=1}^{\infty} a_n$ converges *nonabsolutely*.

Remark. Let $|a_n| \leq b_n$ for each $n \in \mathbf{N}$. If the series $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Theorem 7.4 (limit test)**.** *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with nonnegative members.*

(i) Let
$$\lim_{n\to\infty} \frac{a_n}{b_n}$$
exists finite. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) Let
$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \in (0, +\infty).$$
Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

Theorem 7.5 (Cauchy test)**.** *Let* $\sum_{n=1}^{\infty} a_n$ *be a series. The we have*

(i) If $\lim \sqrt[n]{|a_n|} < 1$*, then* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*
(ii) If $\lim \sqrt[n]{|a_n|} > 1$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Theorem 7.6 (d'Alembert test)**.** *Let* $\sum_{n=1}^{\infty} a_n$ *be a series with nonzero members. Then we have*

(i) If $\lim |a_{n+1}/a_n| < 1$*, then* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*
(ii) If $\lim |a_{n+1}/a_n| > 1$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Theorem 7.7. *Let* $\alpha \in \mathbf{R}$*. The series* $\sum_{n=1}^{\infty} 1/n^{\alpha}$ *converges if and only if* $\alpha > 1$*.*

7.3. **Alternating series.**

Theorem 7.8 (Leibniz)**.** *Let* $\sum_{n=1}^{\infty} (-1)^n a_n$ *be a series. Assume*

- $a_n \geq a_{n+1} \geq 0$ *for every* $n \in \mathbf{N}$*,*
- $\lim_{n \to \infty} a_n = 0$.

Then $\sum_{n=1}^{\infty} (-1)^n a_n$ *converges.*