4.2. **Partial derivatives – basic facts and applications.**

Definition. A *function of n variables* is a mapping defined on a subset of $\mathbf{R}^n$ with values in $\mathbf{R}$.

Definition. Let f be a function of n variables, $j \in \{1, \dots, n\}$, $\boldsymbol{a} \in \mathbf{R}^n$. Then the number

$$\begin{aligned}\frac{\partial f}{\partial x_j}(\boldsymbol{a}) &= \lim_{t\to 0} \frac{f(\boldsymbol{a} + t\boldsymbol{e}^j) - f(\boldsymbol{a})}{t} \\ &= \lim_{t\to 0} \frac{f(a_1, \dots, a_{j-1}, a_j + t, a_{j+1}, \dots, a_n) - f(a_1, \dots, a_n)}{t}\end{aligned}$$

is called *partial derivatives (of first order) of function f according to j-th variable at the point $\boldsymbol{a}$* (if it exists).

Note that $\boldsymbol{e}^j = [0, \dots, 0, \underset{j\text{th coordinate}}{1}, 0, \dots, 0]$.

Remark. Let f be a function of n variables, $\boldsymbol{a} \in \mathbf{R}^n$ and $j \in \{1, \dots, n\}$. Let us define the function

$$\varphi(y) = f(a_1, \dots, a_{j-1}, y, a_{j+1}, \dots, a_n).$$

Then $\varphi'(a_j) = \frac{\partial f}{\partial x_j}(\boldsymbol{a})$ provided one of these derivative exists.

Definition. Let $M \subset \mathbf{R}^n$, $\boldsymbol{x} \in M$, and f be a function defined at least on M, i.e., $M \subset D_f$. We say that f attains at the point $\boldsymbol{x}$

- *maximum on M*, if for every $\boldsymbol{y} \in M$ we have $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$,
- *local maximum with respect to M*, if there exists $\delta > 0$ such that for every $\boldsymbol{y} \in B(\boldsymbol{x}, \delta) \cap M$ we have $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$,
- *sharp local maximum with respect to M*, if there exists $\delta > 0$ such that for every $\boldsymbol{y} \in \big(B(\boldsymbol{x}, \delta) \setminus \{x\}\big) \cap M$ we have $f(\boldsymbol{y}) < f(\boldsymbol{x})$.

The notions *minimum*, *local minimum*, and *sharp local minimum* with respect to M are defined analogically.

Definition. We say that a function f attains at the point $\boldsymbol{x} \in \mathbf{R}^n$ *local maximum*, if $\boldsymbol{x}$ is a local maximum with respect to some ball centered at the point $\boldsymbol{x}$. Similarly one can define *local minimum, sharp local maximum* and *sharp local minimum*.

Theorem 4.6 (necessary condition of existence of local extremum)**.** *Let $G \subset \mathbf{R}^n$ be an open set, $\boldsymbol{a} \in G$, and a function $f\colon G \to \mathbf{R}$ have at the point $\boldsymbol{a}$ local extremum (i.e., a local maximum or a local minimum). Then for each $j \in \{1, \dots, n\}$ we have:*

The partial derivative $\frac{\partial f}{\partial x_j}(\boldsymbol{a})$ either does not exist or is zero.