4.7. **Lagrange multiplier theorem.**

Theorem 4.20 (Lagrange multiplier theorem – basic version). *Let $n \in \mathbf{N}$, $n \geq 2$, let $G \subset \mathbf{R}^n$ be an open set, $f, g \in \mathcal{C}^1(G)$, $M = \{\boldsymbol{x} \in G;\ g(\boldsymbol{x}) = 0\}$, and ley $\tilde{\boldsymbol{x}} \in M$ be a point of local extremum of f with respect to the set M. Then at least one of the following conditions holds:*

(1) $\nabla g(\tilde{\boldsymbol{x}}) = \boldsymbol{o}$,

(2) there exists $\lambda \in \mathbf{R}$ satisfying $\nabla f(\tilde{\boldsymbol{x}}) + \lambda \nabla g(\tilde{\boldsymbol{x}}) = 0$, i.e.m

$$\begin{aligned}
\frac{\partial f}{\partial x_1}(\tilde{x}_1, \dots, \tilde{x}_n) + \lambda \frac{\partial g}{\partial x_1}(\tilde{x}_1, \dots, \tilde{x}_n) &= 0, \\
\frac{\partial f}{\partial x_2}(\tilde{x}_1, \dots, \tilde{x}_n) + \lambda \frac{\partial g}{\partial x_2}(\tilde{x}_1, \dots, \tilde{x}_n) &= 0, \\
&\vdots \\
\frac{\partial f}{\partial x_n}(\tilde{x}_1, \dots, \tilde{x}_n) + \lambda \frac{\partial g}{\partial x_n}(\tilde{x}_1, \dots, \tilde{x}_n) &= 0.
\end{aligned}$$

Theorem 4.21 (Lagrange multiplier theorem – advanced version). *Let $m, n \in \mathbf{N}$, $m < n$, $G \subset \mathbf{R}^n$ be an open set, $f, g_1, \dots, g_m \in \mathcal{C}^1(G)$,*

$$M = \{\boldsymbol{z} \in G;\ g_1(\boldsymbol{z}) = 0, g_2(\boldsymbol{z}) = 0, \dots, g_m(\boldsymbol{z}) = 0\}$$

and let $\tilde{\boldsymbol{z}} \in M$ be a point of local extremum of f with respect to the set M. Then at least one of the following conditions holds:

(1) the vectors

$$\nabla g_1(\tilde{\boldsymbol{z}}), \nabla g_2(\tilde{\boldsymbol{z}}), \dots, \nabla g_m(\tilde{\boldsymbol{z}})$$

are linearly dependent,

(2) there exist $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbf{R}$ satisfying

$$\nabla f(\tilde{\boldsymbol{z}}) + \lambda_1 \nabla g_1(\tilde{\boldsymbol{z}}) + \lambda_2 \nabla g_2(\tilde{\boldsymbol{z}}) + \dots + \lambda_m \nabla g_m(\tilde{\boldsymbol{z}}) = \boldsymbol{o}.$$

Remark. The notion of *linearly dependent vectors* will be defined in the next chapter.

For $m = 1$: One vector is linearly dependent if it is the zero vector.

For $m = 2$: Two vectors are linearly dependent if one of them is a multiple of the other one.