IV. Determine sup and inf of the function f on the set M and decide whether these values are attained

1. $f(x,y,z) = x - 2y + 2z$,
a) $M = \{[x,y,z], x^2+y^2+z^2 = 1\}$, b) $M = \{[x,y,z], x^2+y^2+z^2 = 1, x+y+z = 0\}$

2. $f(x,y,z) = xyz$,
a) $M = \{[x,y,z], x^2+y^2+z^2 = 1\}$, b) $M = \{[x,y,z], x^2+y^2+z^2 = 1,\ x+y+z = 0\}$

3. $f(x,y,z) = \sin x \sin y \sin z$, $M = \{[x,y,z]; x+y+z = \frac{\pi}{2}, x>0, y>0, z>0\}$

4. $f(x_1,\dots,x_n) = x_1^p + \cdots + x_n^p$; $M = \{[x_1,\dots,x_n]; x_1+\cdots+x_n = a,\ x_1>0,\ \dots, x_n>0\}$; where $a>0, p>0$. **5.** $f(x,y,z) = 10z + x - y$, $M = \{[x,y,z], x^2+y^2+z^2 \le 1, y+x \ge 0\}$

6. If $\mathbf{x} \in \mathbf{R}^n$ and $M \subset \mathbf{R}^n$, the distance of the point $\mathbf{x}$ to the set M equals $\operatorname{dist}(\mathbf{x}, M) = \inf\{\rho(\mathbf{x},\mathbf{y})\colon \mathbf{y} \in M\}$, the distance of the set M to a set $N \subset \mathbf{R}^n$ equals $\operatorname{dist}(M,N) = \inf\{\rho(\mathbf{x},\mathbf{y})\colon \mathbf{x} \in M, \mathbf{y} \in N\}$. Compute the distances:
(a) of the point $[a, \frac{1}{2}] \in \mathbf{R}^2$ to the parabola $y = x^2$; (b) of the point $[-a, -\frac{1}{a}] \in \mathbf{R}^2$ $(a>0)$ to the hyperbola branch $y = 1/x$, $x>0$; (c) of the line $y = x - 50$ to the parabola $y = x^2$.

7. $f(x,y) = x^2 + y^2$, $M = \{[x,y], x^2 + 4y^2 = 1\}$

8. If $M \subset \mathbf{R}^n$ is nonempty and bounded, its diameter is defined by $\operatorname{diam} M = \sup\{\rho(\mathbf{x},\mathbf{y})\colon \mathbf{x},\mathbf{y} \in M\}$. Compute the diameter of the set $\{[x,y] \in \mathbf{R}^2\colon |x|^p + |y|^p = 1\}$ (for $p>1$).

Answers and hints. In most problems one can use Lagrange multiplier theorem. **1.** a) max 3 at $[\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}]$, min -3 at $[-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}]$; b) max $\sqrt{\frac{26}{3}}$ at $[\frac{2}{\sqrt{78}}, -\frac{7}{\sqrt{78}}, \frac{5}{\sqrt{78}}]$, min $-\sqrt{\frac{26}{3}}$ at $[-\frac{2}{\sqrt{78}}, \frac{7}{\sqrt{78}}, -\frac{5}{\sqrt{78}}]$; **2.** a) max $\frac{1}{3\sqrt{3}}$ at the points $[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$, $[-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$, $[\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}]$, $[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}]$; min $-\frac{1}{3\sqrt{3}}$ at the points $[-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}]$, $[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$, $[\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$, $[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}]$; b) max $\frac{1}{3\sqrt{6}}$ at the points $[\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}]$, $[-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}]$, $[-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}]$; min $-\frac{1}{3\sqrt{6}}$ at the points $[-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}]$, $[\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}]$, $[\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}]$; **3.** max $\frac{1}{8}$ at $[\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}]$, inf 0, not attained. (Sup and inf on M are the same as on $\overline{M}$.) **4.** for $p=1$ f is constant on M; for $p>1$ sup a^p, not attained, min $\frac{a^p}{n^{p-1}}$ at $[\frac{a}{n}, \dots, \frac{a}{n}]$; for $p \in (0,1)$ inf a^p, not attained, max $\frac{a^p}{n^{p-1}}$ at $[\frac{a}{n}, \dots, \frac{a}{n}]$. (Use the fact that sup and inf on M are the same as on $\overline{M}$, decompose $\overline{M}$ to suitable subsets and use induction n.) **5.** max $\sqrt{102}$ at $[\frac{1}{\sqrt{102}}, -\frac{1}{\sqrt{102}}, \frac{10}{\sqrt{102}}]$; min $-\sqrt{102}$ at $[-\frac{1}{\sqrt{102}}, \frac{1}{\sqrt{102}}, -\frac{10}{\sqrt{102}}]$; **6.** (a) $\sqrt{a^2 - \frac{3}{4}\cdot\sqrt[3]{4}\cdot\sqrt[3]{a^4} + \frac{1}{4}}$ (b) $\frac{1}{a}\sqrt{(1+\sqrt[3]{a^4})^3}$ (c) $\frac{1}{8}\sqrt{79202}$ **7.** max 1 at the points $[\pm 1, 0]$, min $\frac{1}{4}$ at the points $[0, \pm\frac{1}{2}]$ (Geometric interpretation can be used to solve the problem without computation.) **8.** 2 for $p \in (1,2\rangle$, $2^{\frac{3p-2}{2p}}$ for $p>2$. (Use Lagrange multiplier theorem to maximize $\rho([x,y],[u,v])$ for $[x,y]$ and $[u,v]$ from the set. Show that in the maximum point necessarily $x = -u$ and $y = -v$. Further show that for $p \ne 2$ necessarily $x = 0$ or $y = 0$ or $|x| = |y|$. Conclude.)