V. Applications of the implicit function theorem

1. Let us consider the equation $e^{xy} + \sin y + y^2 = 1$ and the point $[2, 0]$:
a) Show that this equation defines a C^∞ function $y = f(x)$ defined on a neighborhood of 2, which satisfies $f(2) = 0$.
b) Determine the tangent line to the graph of f at the point 2.
c) Compute $f''(2)$.
d) For which points $[a, 0]$, $a \in \mathbf{R}$ one can prove an analogue of a) ?
e) Draw the set af all the points $[x, y]$ which satisfy the equation.

2. Consider the equation $x^2 + 2xy^2 + y^4 - y^5 = 0$ and the point $[0, 1]$. Show that
a) this equation defines a C^∞ function $y = f(x)$ in a neighborhood of 0, which satisfies $f(0) = 1$;
b) the function f is strictly increasing on a neighborhood of 0.
c) If f convex or concave on a neighborhood of 0?

3. Let us consider the equation $x^2 + 2y^2 + 3z^2 + xy - z - 9 = 0$ and the point $[1, -2, 1]$:
a) Show that this equation defines a C^∞ function $z = z(x, y)$ on a neighborhood U of $[1, -2]$, which satisfies $z(1, -2) = 1$;
b) compute $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ in U;
c) determine the equation of the tangent plane to the graph of z at the point $[1, -2]$.

4. Show that the set of all the points $[x, y, z] \in \mathbf{R}^3$ satisfying the equation $x^2 + y^2 + z^2 - 3xyz = 0$ can be in a neighborhood of $[1, 1, 1]$ described as the graph of a C^∞ function $f(x, y)$ defined on a neighborhood of $[1, 1]$ which satisfies $f(1, 1) = 1$. Determine the equation of the tangent plane to the graph of f at $[1, 1]$.

Answers and hints. **1.** (b) tangent line: $y = 0$ (c) 0 (d) for $a \neq -1$. (e) The set contains all the x-axis (i.e., points $[a, 0]$ for $a \in \mathbf{R}$), for $a \neq -1$ there is a unique number $\varphi(a) \neq 0$ such that $[a, \varphi(a)]$ belongs to the set. For $a > -1$ one has $\varphi(a) < 0$, for $a < -1$ one has $\varphi(a) > 0$. One can show that $\lim_{a\to-1} \varphi(a) = 0$. **2.** (b) Compute $f'(0)$ and check that $f'(0) = 2 > 0$. (c) We have $f''(0) = -14$, hence $f'' < 0$ on a neighborhood of 0, thus f is strictly concave on a neighborhood of 0. **3.** tangent plane $z = \frac{7}{5}(y + 2) + 1$ **4.** tangent plane $z = -(x - 1) - (y - 1) + 1$