VIII. Antiderivatives and integrals

Compute the following antiderivatives on maximal open intervals

1. $\int x^3+2x+\frac{17}{x}\,\mathrm{d}x$ **2.** $\int 18e^x+16e^{8x}-\frac{1}{x}+3\cos x\,\mathrm{d}x$ **3.** $\int \sqrt[3]{1-3x}\,\mathrm{d}x$ **4.** $\int xe^{-x^2}\,\mathrm{d}x$
5. $\int \operatorname{tg} x\,\mathrm{d}x$ **6.** $\int \operatorname{cotg} x\,\mathrm{d}x$ **7.** $\int \sqrt{x^6}\,\mathrm{d}x$ **8.** $\int \frac{2^{x+1}-5^{x-1}}{10^x}\,\mathrm{d}x$ **9.** $\int \operatorname{tg}^2 x\,\mathrm{d}x$ **10.** $\int \operatorname{cotg}^2 x\,\mathrm{d}x$
11. $\int \sin^2 x\,\mathrm{d}x$ **12.** $\int \cos^4 x\,\mathrm{d}x$ **13.** $\int \frac{x^2\,\mathrm{d}x}{\cos^2(x^3)}$ **14.** $\int \frac{\mathrm{d}x}{x\log x\log\log x}$ **15.** $\int \operatorname{arctg} x\,\mathrm{d}x$
16. $\int \arcsin x\,\mathrm{d}x$ **17.** $\int e^{ax}\cos bx\,\mathrm{d}x$, $a,b\in\mathbb{R}$ **18.** $\int x^\alpha \log x\,\mathrm{d}x$, $\alpha\in\mathbb{R}$
19. $\int x^3\log^2 x\,\mathrm{d}x$ **20.** $\int e^{\sqrt{x}}\,\mathrm{d}x$

Compute the following antiderivatives of rational functions

21. $\int \frac{\mathrm{d}x}{2+3x^2}$ **22.** $\int \frac{x}{1+4x^2}\,\mathrm{d}x$ **23.** $\int \frac{x^{17}-5}{x-1}\,\mathrm{d}x$ **24.** $\int \frac{x^{17}-5}{x^2-1}\,\mathrm{d}x$ **25.** $\int \frac{x^3+1}{x^3-5x^2+6x}\,\mathrm{d}x$
26. $\int \frac{x}{x^3-1}\,\mathrm{d}x$ **27.** $\int \frac{\mathrm{d}x}{x^4+1}$ **28.** $\int \frac{x}{1+x^4}\,\mathrm{d}x$ **29.** $\int \frac{x^8+x-1}{x^6+1}\,\mathrm{d}x$ **30.** $\int \frac{\mathrm{d}x}{(x^2+3x+4)^2}$
31. For which values $a,b,c\in\mathbb{R}$ is an antiderivative of f a rational function, if we have $f(x)=\frac{ax^2+bx+c}{x^3(x-1)^2}$?

Compute the following generalized Riemann integrals

32. $\int_0^2 |1-x|\,\mathrm{d}x$ **33.** $\int_{-1}^1 \frac{\mathrm{d}x}{x^2-2x\cos\alpha+1}$, $\alpha\in(0,\pi)$ **34.** $\int_0^{2\pi}\frac{\mathrm{d}x}{1+\varepsilon\cos x}$, $\varepsilon\in[0,1)$
35. $\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}x}{a^2\sin^2 x+b^2\cos^2 x}$, $ab\neq 0$ **36.** $\int_0^{100\pi}\sqrt{1-\cos 2x}\,\mathrm{d}x$ **37.** $\int_0^{\log 2} xe^{-x}\,\mathrm{d}x$
38. $\int_0^{2\pi} x^2\cos x\,\mathrm{d}x$ **39.** $\int_{\frac{1}{e}}^{e} |\log x|\,\mathrm{d}x$ **40.** $\int_0^{\sqrt{3}} x\operatorname{arctg} x\,\mathrm{d}x$ **41.** $\int_0^{\log 2}\sqrt{e^x-1}\,\mathrm{d}x$
42. $\int_0^a x^2\sqrt{a^2-x^2}\,\mathrm{d}x$ **43.** $\int_0^1 \frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}}\,\mathrm{d}x$ **44.** $\int_{-1}^1\frac{x\,\mathrm{d}x}{x^2+x+1}$ **45.** $\int_1^e (x\log x)^2\,\mathrm{d}x$
46. $\int_0^1 x^{15}\sqrt{1+3x^8}\,\mathrm{d}x$ **47.** $\int_0^{2\pi}\frac{\mathrm{d}x}{(2+\cos x)(3+\cos x)}$ **48.** $\int_0^{2\pi}\frac{\mathrm{d}x}{\sin^4 x+\cos^4 x}$ **49.** $\int_0^\pi e^x\cos^2 x\,\mathrm{d}x$
50. $\int_0^{+\infty} x^n e^{-x}\,\mathrm{d}x$, $n\in\mathbf{N}\cup\{0\}$

Answers and hints. Antiderivatives are written „up to a constant". **1.** $\frac14 x^4+x^2+16\log|x|$ on $(-\infty,0)$ and on $(0,\infty)$; **2.** $18e^x+2e^{8x}-\log|x|+3\sin x$ on $(-\infty,0)$ and on $(0,\infty)$; **3.** $-\frac14(1-3x)^{\frac43}$ on $\mathbb{R}$; **4.** $-\frac12 e^{-x^2}$ on $\mathbb{R}$; **5.** $-\log|\cos x|$ on each of the intervals $(-\frac{\pi}{2}+k\pi,\frac{\pi}{2}+k\pi)$, $k\in\mathbb{Z}$; **6.** $\log|\sin x|$ on each of the intervals $(k\pi,(k+1)\pi)$, $k\in\mathbb{Z}$; **7.** $\frac14|x|\cdot x^3$ on $\mathbb{R}$ (note that $\sqrt{x^6}=|x^3|$; compute antiderivative separately on $(-\infty,0)$ and on $(0,+\infty)$ and then glue the results to get a continuous function); **8.** $\frac{-10\cdot 5^{-x}\log 2+2^{-x}\log 5}{5\log 5\log 2}$ on $\mathbb{R}$; **9.** $\operatorname{tg} x-x$ on each of the intervals $(-\frac{\pi}{2}+k\pi,\frac{\pi}{2}+k\pi)$, $k\in\mathbb{Z}$; **10.** $-\operatorname{cotg} x-x$ on each of the intervals $(k\pi,(k+1)\pi)$, $k\in\mathbb{Z}$; **11.** $\frac{x}{2}-\frac14\sin 2x$ on $\mathbb{R}$; **12.** $\frac38 x+\frac38\sin x\cos x+\frac14\sin x\cos^3 x$ on $\mathbb{R}$; **13.** $\frac13\operatorname{tg}(x^3)$ on each of the intervals $(\sqrt[3]{-\frac{\pi}{2}+k\pi},\sqrt[3]{\frac{\pi}{2}+k\pi})$, $k\in\mathbb{Z}$; **14.** $\log|\log\log x|$ on $(1,e)$ and on (e,∞); **15.** $x\operatorname{arctg} x-\frac12\log(1+x^2)$ on $\mathbb{R}$; **16.** $x\arcsin x+\sqrt{1-x^2}$ on $(-1,1)$; **17.** $\frac{e^{ax}}{a^2+b^2}\cdot(a\cos bx+b\sin bx)$ on $\mathbb{R}$ if $a^2+b^2\neq 0$; x on $\mathbb{R}$ if $a=b=0$; **18.** $\frac{x^{1+\alpha}}{1+\alpha}\cdot\left(\log x-\frac{1}{1+a}\right)$ on $(0,\infty)$ for $\alpha\neq 1$; $\frac12\ln^2 x$ on $(0,\infty)$ for $\alpha=-1$; **19.** $\frac14 x^4\ln^2 x-\frac18 x^4\ln x+\frac{1}{32}x^4$ on $(0,\infty)$; **20.** $2e^{\sqrt{x}}\cdot(\sqrt{x}-1)$ on $(0,+\infty)$ (substitution „$y=\sqrt{x}$"); **21.** $\frac{1}{\sqrt6}\operatorname{arctg}\frac{x\sqrt6}{2}$ on $\mathbb{R}$; **22.** $\frac18\log(1+4x^2)$ on $\mathbb{R}$; **23.** $\left(\sum_1^{17}\frac{x^k}{k}\right)-4\log|x-1|$, $x\in(-\infty,1)$ or $x\in(1,+\infty)$; **24.** $\left(\sum_1^8\frac{1}{2k}x^{2k}\right)-2\log|x-1|+3\log|x+1|$, $x\in(-\infty,-1)$ or $x\in(-1,1)$ or $x\in(1,+\infty)$; **25.** $x+\frac16\log|x|-\frac92\log|x-2|+\frac{28}{3}\log|x-3|$, $x\in(-\infty,0)$ or $x\in(0,2)$ or $x\in(2,3)$ or $x\in(3,+\infty)$; **26.** $\frac16\log\frac{(x-1)^2}{x^2+x+1}+\frac{\sqrt3}{3}\operatorname{arctg}\frac{2x+1}{\sqrt3}$, $x\in(-\infty,1)$ or $x\in(1,+\infty)$; **27.** $\frac{\sqrt2}{8}\log\frac{x^2+x\sqrt2+1}{x^2-x\sqrt2+1}+\frac{\sqrt2}{4}\operatorname{arctg}(x\sqrt2+1)+\frac{\sqrt2}{4}\operatorname{arctg}(x\sqrt2-1)$ on $\mathbb{R}$, (use Moivre theorem to decompose $x^4+1=x^4-(\cos\pi+i\sin\pi)=(x-(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}))(x-(\cos\frac34\pi+i\sin\frac34\pi))(x-(\cos\frac54\pi+i\sin\frac54\pi))(x-(\cos\frac74\pi+i\sin\frac74\pi)))$; **28.** $\frac12\operatorname{arctg}(x^2)$ on $\mathbb{R}$ **29.**

$\frac{1}{3}x^3 + \frac{1}{6}\log(1+x^2) + \frac{\sqrt{3}-1}{2}\log(x^2 - x\sqrt{3}+1) - \frac{\sqrt{3}+1}{2}\log(x^2 + x\sqrt{3}+1) - \frac{3-\sqrt{3}}{6}\operatorname{arctg}(2x-\sqrt{3}) - \frac{3+\sqrt{3}}{6}\operatorname{arctg}(2x+\sqrt{3})$ on $\mathbb{R}$ (use Moivre theorem to decompose $x^6+1 = x^6 - (\cos\pi + i\sin\pi) = (x-(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}))(x-(\cos\frac{3}{6}\pi+i\sin\frac{3}{6}\pi))(x-(\cos\frac{5}{6}\pi+i\sin\frac{5}{6}\pi))(x-(\cos\frac{7}{6}\pi+i\sin\frac{7}{6}\pi))(x-(\cos\frac{9}{6}\pi+i\sin\frac{9}{6}\pi))(x-(\cos\frac{11}{6}\pi+i\sin\frac{11}{6}\pi)))$; **30.** $\frac{1}{7}\cdot\frac{2x+3}{x^2+3x+4} + \frac{4\sqrt{7}}{49}\operatorname{arctg}\frac{2x+3}{\sqrt{7}}$; **31.** if and only if $a+2b+3c=0$; **32.** 1 (decompose to $\int_0^1 + \int_1^2$); **33.** $\frac{\pi}{2\sin\alpha}$; **34.** $\frac{2\pi}{\sqrt{1-\varepsilon^2}}$ (use substitution "$t = \operatorname{cotg}\frac{x}{2}$", i.e., $x = 2\operatorname{arccotg} t$, $t\in\mathbb{R}$, express $\cos x$ using $\operatorname{cotg}\frac{x}{2}$); **35.** $\frac{\pi}{2|ab|}$ (use, for example, substitution $t=\operatorname{tg}x$); **36.** $200\sqrt{2}$ (observe that the integral equals $100\int_0^{\pi}$); **37.** $\frac{1-\log 2}{2}$ **38.** 4π **39.** $2-\frac{2}{e}$ (decompose to $\int_{\frac{1}{e}}^1 + \int_1^e$); **40.** $\frac{2}{3}\pi - \frac{\sqrt{3}}{2}$ **41.** $2-\frac{\pi}{2}$ (use substitution $y = e^x - 1$, i.e., $x = \log(y+1)$); **42.** $\frac{1}{16}\pi a^4$ (use substitution $x = a\sin t$, $t\in(0,\frac{\pi}{2})$); **43.** $\frac{\pi^2}{4}$ (use substitution $y = \arcsin\sqrt{x}$); **44.** $\frac{1}{2}\log 3 - \frac{\sqrt{3}}{6}\pi$ **45.** $\frac{5e^3-2}{27}$ **46.** $\frac{1+\sqrt{2}}{30}$ (use substitution $y = 3x^8$ and then $y = z^2-1$); **47.** $\frac{\pi}{6}(4\sqrt{3}-3\sqrt{2})$ (use the same substitution as in 33) **48.** $2\pi\sqrt{2}$, (show that the integral equals $4\int_0^{\frac{\pi}{2}}$ and use substitution $t = \operatorname{tg}x$ on $(0,\frac{\pi}{2})$); **49.** $\frac{3}{5}(e^{\pi}-1)$; **50.** $n!$ (use integration by parts and induction).