TEST 1, PŘÍKLAD 1 | KRUH 3 |

$a, b \in \mathbb{R} \setminus \{0\}, a \neq b$

$$\lim_{x \to 0} \frac{\cos ax - \cos bx}{\exp ax^2 - \exp bx^2}$$

Najdeme Taylorův rozvoj jmenovatele:

4 body

$\exp(x) = 1 + x + o(x)$ pro $x \to 0$

Tedy $\exp(ax^2) = 1 + ax^2 + o(x^2)$

[zbytková funkce je $\omega(x) \cdot x$, kde $\omega(x) \to 0$ pro $x \to 0$
po dosazení $\underbrace{\omega(ax^2) \cdot a}_{\to 0} x^2$, tedy $o(x^2)$]

Podobně $\exp(bx^2) = 1 + bx^2 + o(x^2)$ pro $x \to 0$

$\Rightarrow \exp(ax^2) - \exp(bx^2) = (a-b)x^2 + o(x^2)$ pro $x \to 0$

Protože čitatel rozvineme do řádu 2:

$$\left.\begin{aligned} \cos ax &= 1 - \frac{a^2x^2}{2} + o(x^2) \\ \cos bx &= 1 - \frac{b^2x^2}{2} + o(x^2) \end{aligned}\right\} \text{pro } x \to 0$$

3 body

Tedy $$\lim_{x \to 0} \frac{\cos ax - \cos bx}{\exp(ax^2) - \exp(bx^2)} = \lim_{x \to 0} \frac{\frac{-a^2+b^2}{2}x^2 + o(x^2)}{(a-b)x^2 + o(x^2)} =$$

$$= \lim_{x \to 0} \frac{\frac{b^2-a^2}{2} + \frac{o(x^2)}{x^2}}{a-b + \frac{o(x^2)}{x^2}} = \frac{\frac{b^2-a^2}{2}}{a-b} = -\frac{a+b}{2}$$

3 body

TEST 1, PŘÍKLAD A1 | KRUH 3 | JINÉ ŘEŠENÍ

$a, b \in \mathbb{R} \setminus \{0\} \quad a \neq b$

$$\lim_{t\to 0} \frac{\cos at - \cos bt}{\exp(at^2) - \exp(bt^2)} \overset{2 \text{ body}}{\underset{\substack{\text{l'H} \\ \text{"}\frac{0}{0}\text{"}}}{=}} \lim_{t\to 0} \frac{-a \sin at + b \sin bt}{\exp(at^2)\cdot 2at - \exp(bt^2)\cdot 2at} =$$

$$\overset{2 \text{ body}}{=} \lim_{t \to 0} \frac{-a \cdot \overbrace{\frac{\sin at}{t}}^{\to a \text{ (*) } 1 \text{ bod}} + b \cdot \overbrace{\frac{\sin bt}{t}}^{\to b \ 1 \text{ bod}}}{\underbrace{\exp(at^2)\cdot 2a - \exp(bt^2)\cdot 2b}_{\to 2a - 2b \ 1 \text{ bod}}} \overset{1 \text{ bod}}{=} \frac{-a^2 + b^2}{2a - 2b} = -\frac{a+b}{2}$$

(*)

$$*) \lim_{t\to 0} \frac{\sin at}{t} = \lim_{t \to 0} \overbrace{\frac{\sin at}{at}}^{\to 1} \cdot a = a$$

$\lim_{y \to 0} \frac{\sin y}{y} = 1$

$\lim_{t \to 0} at = 0$

$t \mapsto at$ je prostá fce

1 bod

(*) nebo jiné ještě jednou l'H "$\frac{0}{0}$":

$$= \lim_{t \to 0} \frac{-a^2 \cos at + b^2 \cos bt}{\exp(at^2)\cdot(2at)^2 + \exp(at^2)\cdot 2a - \exp(bt^2)\cdot(2bt)^2 - \exp(bt^2)\cdot 2b}$$

$$\overset{\text{dosazení}}{=} \frac{-a^2 + b^2}{2a - 2b} = -\frac{a+b}{2}$$

TEST 1, PŘÍKLAD A2 | KRUH 3

$$\lim_{x\to 0} \left(\cosh x^2 - \cos x^2\right)^{\frac{1}{\ln|x|}} \overset{1\text{ bod}}{=} \lim_{x\to 0} \exp\left(\frac{\ln(\cosh x^2 - \cos x^2)}{\ln|x|}\right)$$

dle definice obecné mocniny

Aby to mělo smysl, musí platit $\cosh x^2 - \cos x^2 > 0$ na prstencovém okolí 0. To ovšem platí, protože $\cos x^2 \le 1$ na $\mathbb{R}$ a $\cosh x^2 > 1$
1 bod (chybí-li, −3 body) na $\mathbb{R}\setminus\{0\}$.

Počítejme limitu exponentu:

$$\lim_{x\to 0} \frac{\ln(\cosh x^2 - \cos x^2)}{\ln|x|} \underset{l'H}{\overset{"\frac{-\infty}{-\infty}"}{=}} \lim_{x\to 0} \frac{\frac{1}{\cosh x^2 - \cos x^2}\cdot(\sinh x^2 \cdot 2x + \sin x^2 \cdot 2x)}{\frac{1}{x}} \quad 2\text{ body} =$$

$$= \lim_{x\to 0} \frac{2x^2(\sinh x^2 + \sin x^2)}{\cosh x^2 - \cos x^2} = \underset{(1)}{(*)}$$

Taylorův rozvoj jmenovatele:

$$2\text{ body}\ \begin{cases} \cosh x = 1 + \frac{x^2}{2} + o(x^2) \\ \cos x^2 = 1 - \frac{x^2}{2} + o(x^2) \end{cases} \Rightarrow \begin{aligned} \cosh x^4 &= 1 + \frac{x^4}{2} + o(x^4) \\ \cos x^4 &= 1 - \frac{x^4}{2} + o(x^4) \end{aligned}$$

$$\Rightarrow \cosh x^2 - \cos x^2 = x^4 + o(x^4)$$

Rozvoj čitatele do řádu 4, tedy rozvoj $\sinh x^2 + \sin x^2$ do řádu 2:

$$\begin{aligned} \sinh x &= x + o(x) \\ \sin x &= x + o(x) \end{aligned} \Rightarrow \begin{aligned} \sinh x^2 &= x^2 + o(x^2) \\ \sin x^2 &= x^2 + o(x^2) \end{aligned} \quad 2\text{ body}$$

$$(*) = \lim_{x\to 0} \frac{2x^2\left(x^2 + o(x^2) + x^2 + o(x^2)\right)}{x^4 + o(x^4)} = \lim_{x\to 0} \frac{4 + \frac{o(x^2)}{x^2}}{1 + \frac{o(x^4)}{x^4}} = 4 \quad 1\text{ bod}$$

Tedy původní limita je e^4 1 bod

TEST 1, PŘÍKLAD A2 | KRUH 3 | VARIANTY

$$(1) = \lim_{y\to 0} \frac{2y(\sinh y + \sin y)}{\cosh y - \cos y} = \lim_{y\to 0} \frac{2y^2 \cdot \left(\frac{\sinh y}{y} + \frac{\sin y}{y}\right)}{\cosh y - \cosh y} =$$

VOLSF

$\lim_{x\to 0} x^2 = 0$

$x^2 \neq 0$ pro $x \neq 0$

$$\overset{AL}{=} 2\cdot 2\cdot \lim_{y\to 0} \frac{y^2}{\cosh y - \cosh y} = 4 \cdot \lim_{y\to 0} \frac{y^2}{\left(1+\frac{y^2}{2}+o(y^2)\right) - \left(1-\frac{y^2}{2}+o(y^2)\right)} =$$

$$= 4 \cdot \lim_{y\to 0} \frac{y^2}{y^2 + o(y^2)} = 4 \cdot \lim_{y\to 0} \frac{1}{1+\frac{o(y^2)}{y^2}} = 4 \cdot 1 = 4$$

TEST 1, PŘÍKLAD A2 | ŘRUH3 | VARIANTA

$$\lim_{t\to 0} \left(\cosh t^2 - \cos t^2\right)^{\frac{1}{\ln|t|}} = \lim_{t\to 0} \exp\left(\frac{\ln(\cosh t^2 - \cos t^2)}{\ln|t|}\right)$$

<u>Platí</u>: $\left.\begin{aligned} \cosh t^2 &= 1 + \frac{x^4}{2} + o(t^4) \\ \cos t^2 &= 1 - \frac{t^4}{2} + o(t^4) \end{aligned}\right\}$ pro $t \to 0$

$$\Rightarrow \lim_{t\to 0} \frac{\ln(\cosh t^2 - \cos t^2)}{\ln|X|} = \lim_{t\to 0} \frac{\ln(t^4 + o(t^4))}{\ln|t|} =$$

$$= \lim_{t\to 0} \frac{\ln t^4 (1 + o(1))}{\ln|t|} = \lim_{t\to 0} \frac{\ln t^4 + \ln(1+o(1))}{\ln|t|} =$$

$$= \lim_{t\to 0} \frac{4\ln|t| + \ln(1+o(1))}{\ln|t|} = \lim_{t\to 0} \left(4 + \underbrace{\frac{\overbrace{\ln(1+o(1))}^{\to 0}}{\underbrace{\ln|t|}_{\to -\infty}}}_{\to 0}\right) = 4$$

limita je tedy e^4

TEST 1, PŘÍKLAD A3 | KRUH 3

$$f(z) = \frac{z}{z+2}, \quad z \in \mathbb{C}, \quad \text{střed mocninné řady } 1$$

1. krok: $g(z) := \frac{1}{z+2} = \frac{1}{3+z-1} = \frac{1}{3\left(1+\frac{z-1}{3}\right)} = \frac{1}{3} \cdot \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-1}{3}\right)^n =$

4 body

Součet geometrické řady s kvocientem $-\frac{z-1}{3}$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(z-1)^n}{3^{n+1}}, \quad \text{pro } \left|\frac{z-1}{3}\right| < 1, \text{ neboli } |z-1| < 3$$

1 bod

2. krok: $f(z) = z \cdot g(z) = (z-1)g(z) + g(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n+1}}{3^{n+1}}$

$$+ \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^n}{3^{n+1}} \overset{1 \text{ bod}}{=} \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{(z-1)^n}{3^n} + \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^n}{3^{n+1}} =$$

$$\overset{2 \text{ body}}{=} \frac{1}{3} + \sum_{n=1}^{\infty} \left(\frac{(-1)^{n-1}}{3^n} + \frac{(-1)^n}{3^{n+1}} \right) (z-1)^n = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^{n+1}} (3-1)(z-1)^n$$

$$= \frac{1}{3} + \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{2}{3^{n+1}} (z-1)^n$$

Poloměr konvergence je 3, neboť platí, že

$$\lim_{n \to \infty} \sqrt[n]{\left| (-1)^{n-1} \cdot \frac{2}{3^{n+1}} \right|} = \frac{1}{3} \cdot \sqrt[n]{\frac{2}{3}} \longrightarrow \frac{1}{3} \quad \text{2 body}$$

TEST 1, PŘÍKLAD B3 | DRUH 3, VARIANTA ŘEŠENÍ

$$f(z)=\frac{z}{z+3}=\frac{z+3-3}{z+3}=1-\frac{3}{z+3}=1-\frac{3}{5+(z-2)}=$$

$$=1-\frac{\frac{3}{5}}{1+\frac{z-2}{5}}=1-\frac{3}{5}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n(z-2)^n}{5^n}=$$

pro $\left|\frac{z-2}{5}\right|<1$, tj. $|z-2|<5$

$$=\frac{2}{5}+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{5^{n+1}}(z-2)^n$$

poloměr konvergence je 5, např. protože $\lim_{n\to\infty}\sqrt[n]{\frac{3}{5^{n+1}}}=$

$$=\lim_{n\to\infty}\frac{\sqrt[n]{3}}{5}=\frac{1}{5}$$