TEST 3, KRUH 3, PŘÍKLAD 41

$$\int_{\pi/6}^{\pi/2} \frac{dx}{\sin^4 x}$$ ... použít lze substituci „$y = \operatorname{tg} x$" nebo „$y = \operatorname{cotg} x$" $\Big\}$ 1 bod

Použijeme „$y = \operatorname{cotg} x$"

$\Downarrow$

2 body

$$\int_{\pi/6}^{\pi/2} \frac{1}{\sin^2 x}\cdot\frac{1}{\sin^2 x}\,dx = \int_{\pi/6}^{\pi/2} (\operatorname{cotg}^2 x + 1)\cdot\frac{1}{\sin^2 x}\,dx = (*)$$

$$\left.\begin{aligned} & y = \operatorname{cotg} x, \quad x \in \left(\tfrac{\pi}{6}, \tfrac{\pi}{2}\right) \\ & \frac{dy}{dx} = -\frac{1}{\sin^2 x} < 0 \\ & \operatorname{cotg}\frac{\pi}{2} = 0 \\ & \operatorname{cotg}\frac{\pi}{6} = \frac{\cos\pi/6}{\sin\pi/6} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3} \end{aligned}\right\}$$ 3 body

1 bod

$$(*) = \int_0^{\sqrt{3}} (y^2+1)\,dy = \left[\frac{y^3}{3} + y\right]_0^{\sqrt{3}} = \underbrace{\frac{3\sqrt{3}}{3} + \sqrt{3} - 0 = 2\sqrt{3}}_{2\text{ body}}$$

1 bod

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varianta: „$y = \operatorname{tg} x$" ... 1 bod

$$\int_{\pi/6}^{\pi/2} \frac{1}{\sin^4 x}\cdot\frac{\cos^2 x}{\cos^2 x}\,dx = \int_{\pi/6}^{\pi/2} \frac{1}{(1-\cos^2 x)\cdot\operatorname{tg}^2 x}\cdot\frac{1}{\cos^2 x} =$$

2 body

$$= \int_{\pi/6}^{\pi/2} \frac{1}{\left(1 - \frac{1}{\operatorname{tg}^2 x + 1}\right)\cdot\operatorname{tg}^2 x}\cdot\frac{1}{\cos^2 x}\,dx = \int_{\pi/6}^{\pi/2} \frac{\operatorname{tg}^2 x + 1}{\operatorname{tg}^4 x}\,\frac{1}{\cos^2 x}\,dx = (*)$$

3 body

$$\left[\, y = \operatorname{tg} x, \quad \frac{dy}{dx} = \frac{1}{\cos^2 x} > 0 \quad x \in \left(\tfrac{\pi}{6}, \tfrac{\pi}{2}\right) \Rightarrow y \in \left(\operatorname{tg}\tfrac{\pi}{6}, \infty\right) = \left(\tfrac{1}{\sqrt{3}}, +\infty\right) \right]$$

$$(*) = \int_{1/\sqrt{3}}^{+\infty} \frac{y^2+1}{y^4}\,dy = \int_{1/\sqrt{3}}^{+\infty} \left(\frac{1}{y^2} + \frac{1}{y^4}\right)dy = \underbrace{\left[-\frac{1}{y} - \frac{1}{3y^3}\right]_{1/\sqrt{3}}^{+\infty} = 0 + \sqrt{3} + \frac{(\sqrt{3})^3}{3} = 2\sqrt{3}}_{2\text{ body}}$$

1 bod   1 bod

Príklad B1 $\int_0^{\pi/6} \frac{dx}{\cos^4 x} = \int_0^{\frac{\pi}{6}} \frac{1}{\cos^2 x} \cdot \frac{1}{\cos^2 x} dx = \int_0^{\frac{\pi}{6}} (tg^2 x + 1) \frac{1}{\cos^2 x} dx = (*)$

použijeme substituci

"$y = tg\, x$"

$y = tg\, x, \; x \in (0, \frac{\pi}{6})$

$\frac{dy}{dx} = \frac{1}{\cos^2 x} > 0$

$y \in (0, tg \frac{\pi}{6}) = (0, \frac{1}{\sqrt{3}})$

$tg \frac{\pi}{6} = \frac{\sin \frac{\pi}{6}}{\cos \frac{\pi}{6}} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$

$(*) = \int_0^{1/\sqrt{3}} (y^2 + 1)\, dy = \left[\frac{y^3}{3} + y\right]_0^{1/\sqrt{3}} = \frac{1}{3\sqrt{3} \cdot 3} + \frac{1}{\sqrt{3}} - 0 =$

$= \frac{10}{9\sqrt{3}}$

PŘÍKLAD B2

konvergence $\displaystyle\int_0^{1/3} \frac{x\cos x - \operatorname{arctg} x}{\sqrt{x^7}\cdot\sqrt{1-3x}}\,dx$

• Integrand je spojitý na $(0, \frac{1}{3})$ 1 bod

• v 0+: použijeme Taylorův polynom:

$$x\cos x - \operatorname{arctg} x = x\left(1-\frac{x^2}{2}+o(x^3)\right) - \left(x-\frac{x^3}{3}+o(x^4)\right) = -\frac{1}{6}x^3+o(x^4) \quad \text{pro } x\to 0$$ 2 body

Tedy: $$\lim_{x\to 0+} \frac{\frac{x\cos x-\operatorname{arctg} x}{\sqrt{x^7}\cdot\sqrt{1-3x}}}{\frac{x^3}{\sqrt{x^7}}} = \lim_{x\to 0+}\frac{x\cos x - \operatorname{arctg} x}{x^3\cdot\sqrt{1-3x}} = \lim_{x\to0+}\frac{-\frac{1}{6}x^3+o(x^4)}{x^3\sqrt{1-3x}} = -\frac{1}{6}$$ 1 bod

Tedy: Integrand je záporný na $(0,\delta)$ pro nějaké $\delta>0$ a v 0+ integrál konverguje právě když konverguje $\int_0^{1/6}\frac{x^3}{\sqrt{x^7}}dx = \int_0^{1/6}\frac{1}{\sqrt{x}}dx$. Ten konverguje, proto náš integrál v 0+ konverguje. 2 body

v 1/3 −: Srovnáme s $\frac{1}{\sqrt{1-3x}}$:

$$\lim_{x\to\frac{1}{3}-}\frac{\frac{x\cos x-\operatorname{arctg} x}{\sqrt{x^7}\sqrt{1-3x}}}{\frac{1}{\sqrt{1-3x}}} = \lim_{x\to\frac{1}{3}-}\frac{x\cos x - \operatorname{arctg} x}{\sqrt{x^7}} = \frac{\frac{1}{3}\cos\frac{1}{3}-\operatorname{arctg}\frac{1}{3}}{\sqrt{\frac{1}{3^7}}}\in\mathbb{R}$$ 3 body

limita je vlastní, $\int_{1/6}^{1/3}\frac{1}{\sqrt{1-3x}}dx$ konverguje, tedy integrál v $\frac{1}{3}-$ konverguje

Závěr: Integrál konverguje (absolutně) 1 bod

PŘÍKLAD A2

konvergence integrálu $\displaystyle\int_0^{1/2} \frac{\sin x - \operatorname{arctg} x}{\sqrt{x^7}\,\sqrt{1-2x}}$

- funkce je spojitá na $(0, \frac{1}{2})$

- v $0+$ použijeme Taylorův rozvoj

$$\sin x - \operatorname{arctg} x = x - \frac{x^3}{6} + o(x^4) - \left(x - \frac{x^3}{3} + o(x^4)\right) =$$
$$= \frac{x^3}{6} + o(x^4) \quad \text{pro } x \to 0$$

Srovnáme s $\frac{x^3}{\sqrt{x^7}}$ $\left(= \frac{1}{\sqrt{x}}\right)$

$$\lim_{x\to 0} \frac{\frac{\sin x - \operatorname{arctg} x}{\sqrt{x^7}\,\sqrt{1-2x}}}{\frac{x^3}{\sqrt{x^7}}} = \lim_{x\to 0} \frac{\sin x - \operatorname{arctg} x}{x^3\sqrt{1-2x}} = \lim_{x\to 0} \frac{\frac{x^3}{6} + o(x^4)}{x^3\sqrt{1-2x}} = \frac{1}{6}$$

Tedy integrál v $0+$ konverguje, protože konv. $\int_0^{1/4} \frac{x^3}{\sqrt{x^7}}\,dx = \int_0^{1/4} \frac{1}{\sqrt{x}}\,dx$,

ten konverguje. V $0+$ tedy integrál konverguje

v $1/2-$: Srovnáme s $\frac{1}{\sqrt{1-2x}}$

$$\lim_{x\to 1/2-} \frac{\frac{\sin x - \operatorname{arctg} x}{\sqrt{x^7}\cdot\sqrt{1-2x}}}{\frac{1}{\sqrt{1-2x}}} = \lim_{x\to 1/2-} \frac{\sin x - \operatorname{arctg} x}{\sqrt{x^7}} = \frac{\sin\frac{1}{2} - \operatorname{arctg}\frac{1}{2}}{\sqrt{\frac{1}{2^7}}} \in \mathbb{R}$$

Protože $\int_{1/4}^{1/2} \frac{1}{\sqrt{1-2x}}\,dt$ konverguje, integrál konverguje i v $1/2-$

Závěr: Integrál konverguje (absolutně)

PŘÍKLAD A3

$$f(x,y) = (x+y)^{|x|-|y|} = \exp\big((|x|-|y|)\cdot \lg(x+y)\big)$$

- $D_f = \{ (x,y)\in\mathbb{R}^2 : x+y>0 \}$ 1 bod

- $\frac{\partial f}{\partial x}(x,y) = \exp\big((|x|-|y|)\lg(x+y)\big)\cdot\left(\operatorname{sgn} x \cdot \lg(x+y) + \frac{|x|-|y|}{x+y}\right)$ 1 bod

pro $(x,y)\in D_f$, $x\neq 0$ 1 bod

$\frac{\partial f}{\partial y}(x,y) = \exp\big((|x|-|y|)\lg(x+y)\big)\cdot\left(-\operatorname{sgn} y \cdot \lg(x+y) + \frac{|x|-|y|}{x+y}\right)$ 1 bod

pro $(x,y)\in D_f$, $y\neq 0$ 1 bod

- $\frac{\partial f}{\partial x}(0,y)$, $y\in(0,\infty)$: (*)

Funkce $x\mapsto f(x,y)$ je spojitá na $(-y,+\infty)$, zkusme tedy počítat

$$\frac{\partial f}{\partial x}(0,y) = \lim_{x\to 0}\frac{\partial f}{\partial x}(x,y) = \lim_{x\to 0} \exp\big(\underbrace{(|x|-|y|)\lg(x+y)}_{\to -|y|\cdot \lg y}\big)\cdot\Big(\underbrace{\operatorname{sgn} x}_{\substack{\to 1 \text{ zprava}\\ -1 \text{ zleva}}}\cdot \underbrace{\lg(x+y)}_{\to \lg y} + \underbrace{\frac{|x|-|y|}{x+y}}_{\substack{\to -\operatorname{sgn} y \text{ pro } y\neq 0\\ \text{ale } y>0 \text{ dle } (*)\\ \text{tedy } -1}}\Big)$$

$\to -1$ zleva, 1 zprava pro $y=0$, ale $y>0$ dle (*)

$$= \begin{cases} \text{zleva} & \exp(-|y|\ln y)(-\ln y - 1) \\ \text{zprava} & \exp(-|y|\ln y)(\ln y - 1)\end{cases}$$

3 body

Tedy pro $y\neq 1$ p.d. neexistuje, $\frac{\partial f}{\partial x}(0,1) = -1$

- $\frac{\partial f}{\partial y}(x,0)$, $x\in(0,\infty)$ podobně: fce $y\mapsto f(x,y)$ je spojitá na $(-x,+\infty)$

$$\frac{\partial f}{\partial y}(x,0) = \lim_{y\to 0}\frac{\partial f}{\partial y}(x,y) = \begin{cases} \text{zleva} & \exp(-|x|\ln x)(-\ln x + 1) \\ \text{zprava} & \exp(-|x|\ln x)(\ln x + 1)\end{cases}$$

pro $x\neq 1$ neexistuje

pro $x=1$: $\frac{\partial f}{\partial y}(1,0) = +1$

2 body

Závěr: $\frac{\partial f}{\partial x}(x,y)$
- $x \neq 0,\ y > -x$ .. dáno vzorcem
- $x = 0,\ y = 1$ ... vyjde -1
- $x = 0,\ y \in (0,1) \cup (1,\infty)$ .. neex.

$\frac{\partial f}{\partial y}(x,y)$
- $y \neq 0,\ x > -y$ .. dáno vzorcem
- $y = 0,\ x = 1$ ... vyjde -1
- $y = 0,\ x \in (0,1) \cup (1,\infty)$ .. neex.

PŘÍKLAD B3

$$f(x,y) = (x-y)^{|x|+|y|} = \exp\big((|x|+|y|)\cdot\ln(x-y)\big)$$

- $D_f = \{(x,y)\in\mathbb{R}^2;\ x>y\}$

- $\dfrac{\partial f}{\partial x}(x,y) = \exp\big((|x|+|y|)\ln(x-y)\big)\cdot\left(\operatorname{sgn}(x)\cdot\ln(x-y) + \dfrac{|x|+|y|}{x-y}\right)$

  pro $(x,y)\in D_f,\ x\neq 0$

  $\dfrac{\partial f}{\partial y}(x,y) = \exp\big((|x|+|y|)\ln(x-y)\big)\cdot\left(\operatorname{sgn}(y)\ln(x-y) - \dfrac{|x|+|y|}{x-y}\right)$

  pro $(x,y)\in D_f,\ y\neq 0$

- $\dfrac{\partial f}{\partial x}(0,y),\quad y\in(-\infty,0)$:

  Funkce $x\mapsto f(x,y)$ je spojitá na $(y,+\infty)$, tedy lze počítat

$$\frac{\partial f}{\partial x}(0,y) = \lim_{x\to 0}\frac{\partial f}{\partial x}(x,y) = \lim_{x\to 0}\Big\{\exp\big(\underbrace{(|x|+|y|)\ln(x-y)}_{\downarrow\ |y|\ln(-y)}\big)\cdot$$

$$\cdot\Big(\underbrace{\operatorname{sgn}(x)}_{\substack{\downarrow\\ 1 \text{ zprava}\\ -1 \text{ zleva}}}\underbrace{\ln(x-y)}_{\substack{\downarrow\\ \ln(-y)}} + \underbrace{\frac{|x|+|y|}{x-y}}_{\substack{\downarrow\\ -\operatorname{sgn} y = 1\ (y<0)}}\Big) = \begin{cases}\exp(|y|\ln(-y))(-\ln(-y)+1) & \text{zleva}\\ \exp(|y|\ln(-y))(\ln(-y)+1) & \text{zprava}\end{cases}$$

Tedy pro $y\neq -1$ p.d. neexistuje, $\dfrac{\partial f}{\partial x}(0,-1) = 1$

- $\dfrac{\partial f}{\partial y}(x,0)$ podobně: $x\in(0,\infty)$ $\qquad y\mapsto f(x,y)$ je spojitá na $(-\infty,x)$

$$\frac{\partial f}{\partial y}(x,0) = \lim_{y\to 0}\frac{\partial f}{\partial y}(x,y) = \begin{cases}\exp(|x|\ln x)\,(-\ln x - 1) & \text{zleva}\\ \exp(|x|\ln x)\,(\ln x - 1) & \text{zprava}\end{cases}$$

Tedy $\dfrac{\partial f}{\partial y}(1,0) = -1$, pro $x\neq 1$ p.d. neexistuje