I. INTRODUCTION. ELEMENTS OF MATHEMATICAL LOGIC AND SET THEORY. REAL NUMBERS

Propositional logic.

The language of propositional logic consists of

- a set of primitive symbols (syntactical variables, e.g. A, B, φ, $\psi \ldots$)
- logical operators: $\&$ (in other texts, the mark $\wedge$ is used), $\vee$, $\neg$ ($\sim$ is used as well), $\Rightarrow$, $\Leftrightarrow$
- auxiliary symbols: (,)

Definition 1.1. A *formula* of propositional logic:

(1) Any syntactical variable is a formula (so called *atomic formula*).
(2) If φ and ψ are formulae, then so is $\neg\varphi$ (*negation* of φ), $\varphi \ \& \ \psi$ (*conjunction* of φ and ψ), $\varphi \vee \psi$ (*disjunction* or *alternative* of φ and ψ), $\varphi \Rightarrow \psi$ (*implication*: φ *implies* ψ) and $\varphi \Leftrightarrow \psi$ (*equivalence* of φ and ψ).
(3) Every formula is built inductively from atomic formulae using the previous step.

The truth tables.

1 stands for the truth value *true*, 0 for *false*. The following table shows how to compute truth values of conjunction, disjunction, implication and equivalence of formulas A and B knowing the truth values of A, B.

A	B	$\neg A$	$A \ \& \ B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
1	1	0	1	1	1	1
1	0	0	0	1	0	0
0	1	1	0	1	1	0
0	0	1	0	0	1	1

Proposition 1.1. *Any implication is equivalent to its contrapositive:*

$$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A).$$

Proposition 1.1 gives an important method of proving implications (see below).

Proposition 1.2 (De Morgan Laws).

$$\neg(A \ \& \ B) \Leftrightarrow (\neg A \vee \neg B)$$
$$\neg(A \vee B) \Leftrightarrow (\neg A \ \& \ \neg B)$$

A formula which is true regardless of the truth values of its atomic subformulas is called *tautology.*

First order logic (predicate calculus).

Mathematical theories are expressed using first order logic. It differs from propositional logic by its use of quantified variables: $(\forall x \in A)$, $(\forall y \in \mathbb{R})$, $(\forall f_1 \in F)$ etc.; we often write $(\forall x)$, $(\forall n)$, when it is clear to which set x (n, respectively) belongs. Further, language of first order logic involves function symbols (e.g., $+$, . are binary function symbols, $\sqrt{\ }$ is unary) and predicate symbols (e.g., $<$, $\leq$, $\in$, $=$, $\subseteq$ and $\neq$ are binary predicate symbols). Function symbols represent operations with mathematical objects, e.g. with numbers and functions, predicate symbols represent relations between such objects. Use of logical operators and auxiliary symbols remains the same as in propositional logic. *Formulae* are built inductively using logical operators and quantifying over individual variables.

Sets.

A *set* is a collection of well defined and distinct objects. E.g., $\mathbb{R}$ is the set of all real numbers, $\mathbb{N}$ is the set of all natural numbers. Another example is a set of points in the plane with given property or the set of all functions with nonnegative values.

Notation and terminology.

- $x \in A$ . . . x is *an element* of A, x belongs to A
- $x \notin A$ . . . x is not an element of A
- $A \subseteq B$ or $A \subset B$ . . . the set A is *a subset* of B (*inclusion*)
- $A = B$ . . . the sets A and B have the same elements, equivalently $A \subset B$ & $B \subset A$
- $\emptyset$ . . . *the empty set,* $\emptyset = \{x;\ x \neq x\}$
- $A \cup B$ . . . *union* of the sets A and B; $A \cup B = \{x;\ x \in A \vee x \in B\}$
- $A \cap B$ . . . *intersection* of the sets A and B; $A \cap B = \{x;\ x \in A\ \&\ x \in B\}$
- $A \cap B = \emptyset$ . . . A and B are *disjoint*, i.e. they have no common element
- $A \setminus B$ . . . relative *complement*, set theoretic *difference* of the sets A and B; $A \setminus B = \{x \in A;\ x \notin B\}$; e.g. $\mathbb{R} \setminus \mathbb{Q} = \mathbb{I}$ – the set of irrationals
- $A_1 \times A_2 \times \cdots \times A_n = \big\{[a_1, \dots, a_n];\ a_1 \in A_1\ \&\ \cdots\ \&\ a_n \in A_n\big\}$ . . . *Cartesian product* of the the sets $A_1, \dots, A_n$

Let I be a nonempty set of indices, and let A_α be a set for each $\alpha \in I$.

- $\bigcup_{\alpha \in I} A_\alpha = \{x;\ (\exists \alpha \in I)\ x \in A_\alpha\}$ . . . *union* of all $A_\alpha s$, the set of all elements belonging to at least one A_α
- $\bigcap_{\alpha \in I} A_\alpha = \{x;\ (\forall \alpha \in I)\ x \in A_\alpha\}$ . . . *intersection* of all $A_\alpha s$, the set of all elements belonging to A_α for every α

Formulae with quantifiers.

With given set M and formula φ (assertion about x),

- $(\forall x \in M)\, \varphi(x)$ means 'for all $x \in M$ $\varphi(x)$ holds',
- $(\exists x \in M)\, \varphi(x)$ means 'there is at least one $x \in M$ such that $\varphi(x)$ holds',
- $(\exists! x \in M)\, \varphi(x)$ is an abbreviation for 'there exists a unique (i.e. exactly one) $x \in M$ such that $\varphi(x)$ holds'.

Notice that

- $(\forall x \in M)\, \varphi(x)$ means $(\forall x)\, (x \in M \Rightarrow \varphi(x))$,
- $(\exists x \in M)\, \varphi(x)$ means $(\exists x)\, (x \in M\ \&\ \varphi(x))$.

Negation of formulae with quantifiers.

- $\neg(\forall x \in M)\, \varphi(x)$ is the same as $(\exists x \in M)\, \neg\varphi(x)$,
- $\neg(\exists x \in M)\, \varphi(x)$ is the same as $(\forall x \in M)\, \neg\varphi(x)$.

Proof methods.
• **Direct proof.** The aim is to prove $A \Rightarrow B$. We do it via proving sequence of assertions: $A \Rightarrow C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_n \Rightarrow B$.
• **Indirect proof.** Proving a contrapositve $\neg B \Rightarrow \neg A$ instead of $A \Rightarrow B$.
• **Proof by contradiction.** We want to prove an assertion A. To this aim, we assume $\neg A$ holds and deduce (in several steps) a contradiction from it. I.e. a formula of the form $B \ \& \ \neg B$ (typically $A \ \& \ \neg A$). Since we have come to a nonsense, the assumption $\neg A$ was wrong and A holds true.
• **Mathematical induction.** We want to prove an assertion $\varphi(n)$ for every natural number n (i.e. $n = 1, 2, 3, \dots$). First, we prove $\varphi(1)$. Then, assuming $\varphi(n)$ holds (such an assumption is called *inductive assumption*), we prove $\varphi(n+1)$. Since n has been chosen arbitrarily, we can conclude $\varphi(n)$ holds for every natural n.

Theorem 1.3 (De Morgan Laws for sets). *Suppose $I \neq \emptyset$, S, A_α (for every $\alpha \in I$) are sets. Then*

$$S \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (S \setminus A_\alpha),$$
$$S \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (S \setminus A_\alpha).$$

Sketch of the proof. The equalities can be proven directly (regarding the notion of equality): suppose x is an element of the left hand side and show it is also an element of the right hand side. And vice versa. □

Example. Irrationality of $\sqrt{2}$. *$\sqrt{2} \notin \mathbb{Q}$, i.e. if $x \in \mathbb{R}$ and $x^2 = 2$, then x is irrational.*

Proof. Assume for contradiction $\sqrt{2} = x = \frac{p}{q}$ so that p, $q \in \mathbb{N}$ and the fraction is *irreducible* (i.e. p, q are *coprime*). Hence $\frac{p^2}{q^2} = x^2 = 2$, it means $p^2 = 2q^2$ and, consequently, p is even. Let $p = 2r$, $r \in \mathbb{N}$. Then $p^2 = 4r^2 = 2q^2$, thus $2r^2 = q^2$ and q is even as well. This contradicts assumption on p, q being coprime. So, the assumption that x can be expressed as $\frac{p}{q}$ was false, and x is irrational. □

Example. $(\forall n \in \mathbb{N})\ 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Proof. Easily by induction on n. □

Example. Binomial theorem. For every a, $b \in \mathbb{R}$ and every $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Recall that for natural numbers n and k, $n \geq k$, the *binomial coefficient*

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot (n-1) \dots (n-k+1)}{k!}$$

is equal to the number of k-element subsets of an n-element set. Further $\binom{n}{0} = \binom{n}{n} = 1$, of course, $\binom{n}{1} = n$. In the proof, the equality

$$\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}$$

uses. Finally, $\sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n$.

Proof of binomial theorem. By induction on n. For $n = 1$, the left hand side, L, equals $a+b$, while the right hand side $R = \binom{1}{0} a^0 b^1 + \binom{1}{1} a^1 b^0 = b + a$ ($k \in \{0; 1\}$ here). Hence $L = R$.

Suppose binomial theorem holds for given n and let us prove it for $n+1$. $L = (a+b)^{n+1} = (a+b) \cdot (a+b)^n$. According to the induction assumption, the latter equals to

$$(a+b) \cdot \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1}.$$

Now, let us rename the variable k in both sums: we put $j = k+1$ in the first sum and $j = k$ in the second sum, respectively. Thus

$$\begin{aligned} L = \sum_{j=1}^{n+1} \binom{n}{j-1} a^j b^{n-j+1} + \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j+1} \\ = \sum_{j=1}^{n} \left(\binom{n}{j-1} + \binom{n}{j} \right) a^j b^{n-j+1} + \binom{n}{0} a^0 b^{n+1} + \binom{n}{n} a^{n+1} b^0. \end{aligned}$$

According to the equalities mentioned above, this equals the right hand side for $n+1$. $\square$

Sets of numbers.

- The set of all *natural numbers* $\mathbb{N} = \{1, 2, 3, \dots\}$.
- The set of *integers* $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n;\, n \in \mathbb{N}\} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.
- The set of all *rational numbers* $\mathbb{Q} = \left\{ \frac{p}{q};\, p \in \mathbb{Z} \;\;\&\;\; q \in \mathbb{N} \right\}$, where

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} \Leftrightarrow p_1 q_2 = p_2 q_1.$$

Definition 1.2. The set of all *real numbers* $\mathbb{R}$ is a set with operations $+$ (*addition*) and $\cdot$ (*multiplication*) and with relation $\leq$ (*ordering*), in which the following three groups of axioms are satisfied:
I. properties of addition and multiplication and their relationships,
II. properties of ordering and its relationships to addition and multiplication,
III. Infimum Axiom.

Once we specify the groups of properties I, II, III, Definition 1.2 will be complete.

I. Properties of addition and multiplication.

- $+$ and $\cdot$ are *commutative*: $(\forall x, y \in \mathbb{R})\, (x + y = y + x \;\;\&\;\; x \cdot y = y \cdot x)$
- $+$ and $\cdot$ are *associative*: $(\forall x, y, z \in \mathbb{R})\, \big(x + (y + z) = (x + y) + z \;\;\&\;\; x \cdot (y \cdot z) = (x \cdot y) \cdot z\big)$

- existence of *additive identity element, zero*: $(\exists 0 \in \mathbb{R})\,(\forall x \in \mathbb{R})\, x + 0 = x$
- existence of *multiplicative identity element, one*: $(\exists 1 \in \mathbb{R})\big(0 \neq 1 \ \ \& \ \ (\forall x \in \mathbb{R})\,(x \neq 0 \Rightarrow x \cdot 1 = x)\big)$
- *distributivity* of $\cdot$ over $+$: $(\forall x, y, z \in \mathbb{R})\, x \cdot (y + z) = x \cdot y + x \cdot z$
- existence of *additive inverses*: $(\forall x \in \mathbb{R})\,(\exists -x \in \mathbb{R})\, x + (-x) = 0$
- existence of *multiplicative inverses*: $(\forall x \in \mathbb{R})\,\big(x \neq 0 \Rightarrow (\exists x^{-1} \in \mathbb{R})\, x \cdot x^{-1} = 1\big)$

II. Properties of ordering.

- $\leq$ is *transitive*: $(\forall x, y, z \in \mathbb{R})\,\big((x \leq y \ \ \& \ \ y \leq z) \Rightarrow x \leq z\big)$
- $\leq$ is *weakly antisymmetric*: $(\forall x, y \in \mathbb{R})\,\big((x \leq y \ \ \& \ \ y \leq x) \Rightarrow x = y\big)$
- *dichotomy* of $\leq$: $(\forall x, y \in \mathbb{R})\,\big(x \leq y \vee y \leq x\big)$ (any two elements can be compared)
- $(\forall x, y, z \in \mathbb{R})\,\big(x \leq y \Rightarrow x + z \leq y + z\big)$
- $(\forall x, y \in \mathbb{R})\,\big((0 \leq x \ \ \& \ \ 0 \leq y) \Rightarrow 0 \leq x \cdot y\big)$

Definition 1.3. We say that a set $M \subset \mathbb{R}$ is *bounded from below*, if there exists a number $a \in \mathbb{R}$ such that $(\forall x \in M)\, a \leq x$. Such a is called *lower bound of M*. Analogously, we define sets *bounded from above* and *upper bound.* We say that a set $M \subset R$ is *bounded* if it is bounded from below and from above.

III. **Infimum Axiom.** Let M be a nonempty set bounded from below. Then there exists a unique number $i \in \mathbb{R}$ with the following properties:

(i) $(\forall x \in M)\, x \geq i$ (i.e. i is a lower bound of M),
(ii) $(\forall i' \in \mathbb{R})\,\big(i' > i \Rightarrow (\exists x \in M)\, x < i'\big)$ (i.e. i is the biggest lower bound).

Definition 1.4. The $i \in \mathbb{R}$ from Infimum Axiom is called *infimum of M* and denoted $i = \inf M$. Analogously, for a nonempty set N bounded from above, *supremum of N*, $\sup N$ defines as the least upper bound of N.

Remarks.

(1) The set $\mathbb{R}$ exists and is defined uniquely by I–III (a deep theorem of set theory).
(2) $(\forall x \in \mathbb{R})\, 0 \cdot x = x \cdot 0 = 0$ (0 makes every element vanish).
(3) $(\forall x \in \mathbb{R})\, -x = (-1) \cdot x$
(4) $(\forall x \in \mathbb{R} \setminus \{0\})\,(\forall n \in \mathbb{N})\, x^{-n}$ can be defined as $(x^n)^{-1} = (x^{-1})^n$ (x^n defines inductively: $x^1 = x$, $x^{n+1} = x^n \cdot x$ for every $x \in \mathbb{R}$).
(5) $(\forall x \in \mathbb{R})\,(\forall y \in \mathbb{R})\,\big((x > 0 \ \ \& \ \ y > 0) \Rightarrow x \cdot y > 0\big)$

Intervals.

Let a, $b \in \mathbb{R}$, $a < b$.

- $(a, b) = \{x \in \mathbb{R};\ a < x < b\}$ . . . *open interval*
- $\langle a, b\rangle = \{x \in \mathbb{R};\ a \leq x \leq b\}$ . . . *closed interval*
- $(a, b\rangle = \{x \in \mathbb{R};\ a < x \leq b\}$ . . . *semiopen interval*
- $\langle a, b) = \{x \in \mathbb{R};\ a \leq x < b\}$ . . . *semiclosed interval*
- *unbounded intervals:*

$$(a, +\infty) = \{x \in \mathbb{R};\ a < x\}$$
$$(-\infty, a) = \{x \in \mathbb{R};\ a > x\}$$
$$\langle a, +\infty) = \{x \in \mathbb{R};\ a \leq x\}$$
$$(-\infty, a\rangle = \{x \in \mathbb{R};\ a \geq x\}$$

Theorem 1.4 (Supremum Theorem). *If $M \subset \mathbb{R}$ is nonempty bounded from above then there exists the unique $s = \sup M$.*

Sketch of the proof. $\sup M = -\inf(-M)$, where $-M = \{x \in \mathbb{R};\ -x \in M\}$. $\square$

Definition 1.5. Let $M \subset \mathbb{R}$. We call $a \in \mathbb{R}$ *maximum of* M and write $a = \max M$ if $a \in M$ and $(\forall x \in M)\, x \leq a$. *Minimum of* M, $\min M$, defines analogously.

Theorem 1.5 (Existence of integer part).

$$(\forall x \in \mathbb{R})\, (\exists! k \in \mathbb{Z})\, k \leq x < k+1.$$

Such k is called *integer part* of x and is denoted by $[x]$ or $\lfloor x \rfloor$.

Proof. Let $x \in \mathbb{R}$ and put $M = \{n \in \mathbb{Z};\ n \leq x\}$. Clearly, the set is bounded from above. We shall prove M is nonempty.

Assume for contradiction $M = \emptyset$. From dichotomy of $\leq$, it follows that $(\forall z \in \mathbb{Z})\, z > x$. Under the assumption, the set $\mathbb{Z}$ is nonempty and bounded from below, hence by Infimum Axiom there exists $y = \inf \mathbb{Z} \in \mathbb{R}$. It follows that $(\forall z \in \mathbb{Z})\, z - 1 \geq y$, i.e., $(\forall z \in \mathbb{Z})\, z \geq y + 1$ – a contradiction with definition of infimum. We can conclude that $M \neq \emptyset$.

Now we are ready to use Supremum Theorem: let $G = \sup M$. Since G is the lowest upper bound of M, $G-1$ is not an upper bound, thus there is $k \in M$ such that $G-1 < k$. I.e. $G < k+1$ which is therefore not in M. Hence $k \leq x < k+1$ for such k. $\square$

Theorem 1.6 (Archimede Property).

$$(\forall x \in \mathbb{R})\, (\exists n \in \mathbb{N})\, x \leq n.$$

Proof. Put $n = \max\{1, \lfloor x \rfloor + 1\}$. $\square$

Theorem 1.7 (Existence of n-th root).

$$(\forall x \in \langle 0, \infty))\, (\forall n \in \mathbb{N})\, (\exists! y \in \langle 0, \infty))\, y^n = x.$$

Theorem 1.8 (Density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$). *Let a, $b \in \mathbb{R}$, $a < b$. Then*

$$(\exists q \in \mathbb{Q})\, a < q < b,$$
$$(\exists j \in \mathbb{R} \setminus \mathbb{Q})\, a < j < b.$$

II. SEQUENCES. LIMITS

Definition 2.1. An assignment $n \mapsto a_n$, where $a_n \in \mathbb{R}$ for each $n \in \mathbb{N}$, is called *sequence*, ve denote it $\{a_n\}_{n=1}^{\infty}$ or simply $\{a_n\}$; a_n is called *n-th member of the sequence* $\{a_n\}_{n=1}^{\infty}$. Two sequences are equal, $\{a_n\}_{n=1}^{\infty} = \{b_n\}_{n=1}^{\infty}$ iff $(\forall n \in \mathbb{N})\, a_n = b_n$.

Definition 2.2. We say that a sequence $\{a_n\}_{n=1}^{\infty}$ is *bounded from below* (*bounded from above*, *bounded*, respectively), if the corresponding set $\{a_n;\, n \in \mathbb{N}\}$ is bounded from below (bounded from above, bounded, resp.).

Definition 2.3 (Monotonicity). We say that $\{a_n\}_{n=1}^{\infty}$ is

- *increasing*, if $(\forall n \in \mathbb{N})\, a_n < a_{n+1}$
- *decreasing*, if $(\forall n \in \mathbb{N})\, a_{n+1} < a_n$
- *nonincreasing*, if $(\forall n \in \mathbb{N})\, a_{n+1} \leq a_n$
- *nondecreasing*, if $(\forall n \in \mathbb{N})\, a_n \leq a_{n+1}$
- *monotone*, if it is nonincreasing or nondecreasing
- *strictly monotone*, if it is increasing or decreasing

Examples.

(1) $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$ is decreasing.
(2) $\{a_n\}_{n=1}^{\infty}$, where $a_n = n$ for each $n \in \mathbb{N}$, is increasing.
(3) *Fibonacci sequence* $(0, 1, 1, 2, 3, 5, 8, 13, \dots)$, i.e. $a_1 = 0$, $a_2 = 1$, $a_{n+2} = a_n + a_{n+1}$, is nondecreasing. It is a subsequence (see Definition 2.5 below) of the previous sequence.
(4) $\{(-1)^n\} = (-1, 1, -1, 1, -1, \dots)$ is not monotone.

Definition 2.4. We say that a sequence $\{a_n\}_{n=1}^{\infty}$ has *limit* $A \in \mathbb{R}$ (or *converges to* A), if

$$(\forall \varepsilon \in \mathbb{R}, \varepsilon > 0)\, (\exists n_0 \in \mathbb{N})\, (\forall n \in \mathbb{N}, n \geq n_0)\ |a_n - A| < \varepsilon.$$

We shall denote this fact by

$$\lim_{n\to\infty} a_n = A, \text{ or briefly } \lim a_n = A,\ a_n \xrightarrow[n\to\infty]{} A, \text{ or } a_n \to A.$$

A sequence $\{a_n\}$ is called *convergent* if it has a limit $A \in \mathbb{R}$.

Example. A useful limit is

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

This proves from the definition of limit using Archimede Property (Theorem 1.6).

Theorem 2.1. *Each sequence has at most one limit.*

Theorem 2.2. *Each convergent sequence is bounded.*

Definition 2.5. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that $\{b_k\}_{k=1}^{\infty}$ is a *subsequence* of $\{a_n\}_{n=1}^{\infty}$, if there is an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $(\forall k \in \mathbb{N})\, b_k = a_{n_k}$.

Theorem 2.3 (Subsequences preserve limits). *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence,* $\{b_k\}_{k=1}^{\infty}$ *its subsequence,* $\lim_{n\to\infty} a_n = A$*. Then* $\lim_{k\to\infty} b_k = A$*.*

Example. $\{(-1)^n\}$ has no limits. Indeed, its subsequence $\{(-1)^{2k}\}_{k=1}^{\infty} = (1, 1, 1, \dots)$ has limit 1, while another subsequence $\{(-1)^{2k+1}\}_{k=1}^{\infty} = (-1, -1, -1, \dots)$ converges to -1.

Definition 2.6. For sequences $\{a_n\}$, $\{b_n\}$ and a constant $\lambda \in \mathbb{R}$, we define

- $\{a_n\} + \{b_n\} = \{a_n + b_n\}$
- $\{a_n\} \cdot \{b_n\} = \{a_n \cdot b_n\}$
- $\lambda \cdot \{a_n\} = \{\lambda \cdot a_n\}$
- if $(\forall n \in \mathbb{N})\ b_n \neq 0$, then $\frac{\{a_n\}}{\{b_n\}} = \left\{\frac{a_n}{b_n}\right\}$

Theorem 2.4 (Arithmetics of limits). *Let* $\lim a_n = A \in \mathbb{R}$*,* $\lim b_n = B \in \mathbb{R}$*. Then*

(1) $\lim(a_n + b_n) = A + B$
(2) $\lim(a_n \cdot b_n) = A \cdot B$
(3) *if, moreover,* $b_n \neq 0$ *for each* n *and* $B \neq 0$ *then* $\lim \frac{a_n}{b_n} = \frac{A}{B}$

Definition 2.7. For $a \in \mathbb{R}$ the *absolute value* of a defines

$$|a| = \begin{cases} a \text{ if } a \geq 0, \\ -a \text{ if } a < 0. \end{cases}$$

Proposition 2.5 (The Triangle Inequality). *For every* x*,* $y \in \mathbb{R}$*,*

$$|x + y| \leq |x| + |y|.$$

Example.

$$\lim_{n\to\infty} \frac{3n^2 - 5n + 11}{2n^2 + 4n - 7} = \lim_{n\to\infty} \frac{3 - \frac{5}{n} + \frac{11}{n^2}}{2 + \frac{4}{n} - \frac{7}{n^2}} = \frac{3 - 0 + 0}{2 + 0 - 0} = \frac{3}{2},$$

because $\lim \frac{1}{n} = 0$ and arithmetics of limits can be applied.

Theorem 2.6. *Let* $a_n \to 0$*, let* $\{b_n\}$ *be a bounded sequence. Then* $a_n \cdot b_n \to 0$*.*

Theorem 2.7 (Limits preserve ordering). *Let* $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$*.*
(i) Suppose there exists $n_0 \in \mathbb{N}$ *such that* $a_n \leq b_n$ *for each* $n \geq n_0$*. Then* $A \leq B$*.*
(ii) Let $A < B$*. Then there exists* $n_0 \in \mathbb{N}$ *such that* $a_n < b_n$ *for each* $n \geq n_0$*.*

Example. $\leq$ cannot be replaced by $<$ in the previous theorem: consider the sequences $\left\{-\frac{1}{n}\right\}$ and $\left\{\frac{1}{n}\right\}$ – they have common limit 0.

Theorem 2.8 (Two Policemen or **Sandwich Theorem).** *Let* $\{a_n\}$*,* $\{b_n\}$ *be convergent sequences and let* $\{c_n\}$ *be a sequence such that*
(i) $(\exists n_0 \in \mathbb{N})\,(\forall n \geq n_0)\, a_n \leq c_n \leq b_n$ *and*
(ii) $\lim a_n = \lim b_n = A$*.*
Then $\{c_n\}$ *is convergent and* $\lim c_n = A$*.*

Definition 2.8. We say that a sequence $\{a_n\}$ has *limit* $+\infty$ if

$$(\forall L \in \mathbb{R})\,(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N})\,a_n > L.$$

We say that a sequence $\{a_n\}$ has *limit* $-\infty$ if

$$(\forall K \in \mathbb{R})\,(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N})\,a_n < K.$$

The structure of the real line now extends by adding two limit elements, $+\infty$ and $-\infty$:

$$\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}.$$

We have to define ordering $<$ and operations $+$ and $\cdot$ on this extended structure.

- $(\forall a \in \mathbb{R})\; -\infty < a < +\infty$
- $(\forall a \in \mathbb{R})\; \pm\infty + a = a + \pm\infty = \pm\infty$
- $(\forall a \in \mathbb{R}^* \setminus \{0\})\Big(+\infty \cdot a = a \cdot +\infty = \begin{cases} +\infty \text{ if } a > 0 \\ -\infty \text{ if } a < 0 \end{cases}$

  $\& \;\; -\infty \cdot a = a \cdot -\infty = \begin{cases} -\infty \text{ if } a > 0 \\ +\infty \text{ if } a < 0 \end{cases}\Big)$
- $\pm\infty + \pm\infty = \pm\infty$
- $(\forall a \in \mathbb{R})\; \frac{a}{\pm\infty} = 0$

Important remark. The following expressions are not defined:

$$\text{"}\infty - \infty\text{"},\ \text{"}0 \cdot \pm\infty\text{"},\ \text{"}\frac{a}{0}\text{"},\ \text{"}\frac{\infty}{\infty}\text{"}.$$

Practically, it means that if arithmetics of limits produces such an expressesion, other methods have to be used to compute the limit, including, of course, rearrangement of the original expression.

Example. $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$ leads to "$\infty - \infty$" at the first sight (because $n+1$ and n as well as their square roots have limit $+\infty$). Let us rearrange the problem:

$$\sqrt{n+1} - \sqrt{n} = \big(\sqrt{n+1} - \sqrt{n}\big) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0,$$

because $\sqrt{n+1} + \sqrt{n} \to +\infty$.

Theorem 2.9 (Arithmetics of limits). *Let* $\lim a_n = A \in \mathbb{R}^*$ *and* $\lim b_n = B \in \mathbb{R}^*$. *Then*
(i) $\lim(a_n + b_n) = A + B$, *if the right side is defined,*
(ii) $\lim(a_n \cdot b_n) = A \cdot B$, *if the right side is defined,*
(iii) $\lim \frac{a_n}{b_n} = \frac{A}{B}$, *if the right side is defined.*

Remark. Each sequence of real numbers has at most one limit in $\mathbb{R}^*$. Limits preserve $\leq$ in $\mathbb{R}^*$ and an obvious modification of Sandwich Theorem holds.

Theorem 2.10. *Each monotone sequence has a limit.*

Example. $\lim \left(1 + \frac{1}{n}\right)^n$ defines $e \approx 2.71$. To see that the sequence is convergent, it suffices to show it is increasing and bounded from above; both facts are non-obvious. Detailed comment later.

Theorem 2.11 (Bolzano–Weierstrass). *Each bounded sequence contains a convergent subsequence.*

III. FUNCTIONS

Definition 3.1. Let A and B be nonempty sets. A *mapping* is an assignment

$$\begin{aligned} f : A &\to B, \\ x &\mapsto f(x), \end{aligned}$$

such that $(\forall x \in A)\,(\exists! y \in B)\, y = f(x)$.

Definition 3.2. Let $f : A \to B$ be a mapping. Its *domain* defines as $\mathcal{D}_f = A$, its *range* $\mathcal{R}_f = \{f(x);\, x \in A\}$. For $X \subset A$, *image of* X is $f[X] = \{f(x);\; x \in A\}$, for $Y \subset B$, *preimage of* Y equals $f^{-1}[Y] = \{x \in A;\; (\exists y \in Y) f(x) = y\}$.

The *graph of* f is defined as $\mathcal{G}_f = \{[x, y] \in A \times B;\; y = f(x)\}$.

Definition 3.3. A mapping $f : A \to B$ is *onto* if $\mathcal{R}_f = B$. It is *injective* or *one-to-one* if

$$(\forall x_1, x_2 \in A)\,\big(f(x_1) = f(x_2) \Rightarrow x_1 = x_2\big).$$

A mapping is called *bijective* if it is injective and onto.

Let $f : A \to B$, $g : B \to C$ be mappings. The symbol $g \circ f$ stands for their *composition*, i.e. a mapping from A to C defined by

$$(g \circ f)(x) = g(f(x)),\; x \in A.$$

Let $f : A \to B$ be injective and onto. *Inversion mapping* $f^{-1} : B \to A$ is defined by $f^{-1}(y) = x$, where x satisfies $f(x) = y$.

Definition 3.4. A mapping f is a *function of one real variable* (a *function* for short) if $f : M \to \mathbb{R}$, where $M \subset \mathbb{R}$.

Definition 3.5. A function $f : J \to \mathbb{R}$ is *increasing* on an interval J, if for each pair x_1, $x_2 \in J$, $x_1 < x_2$, the inequality $f(x_1) < f(x_2)$ holds. The notions of *decreasing*, *nondecreasing*, *nonincreasing functions* are defined in an analogous way.

By *monotone function* (*strictly monotone function*, respectively) on the interval J we mean a function, which is nondecreasing or nonincreasing (increasing or decreasing respectively) on J.

Definition 3.6. We say that a function $f : \mathcal{D}_f \to \mathbb{R}$ is

- *odd*, if for each $x \in \mathcal{D}_f$, $-x \in \mathcal{D}_f$ and $f(-x) = -f(x)$,
- *even*, if for each $x \in \mathcal{D}_f$, $-x \in \mathcal{D}_f$ and $f(-x) = f(x)$.

Definition 3.7. A function $f : \mathcal{D}_f \to \mathbb{R}$ is called *periodic* with *period* $a \in \mathbb{R}$, $a > 0$, if for each $x \in \mathcal{D}_f$, $x + a \in \mathcal{D}_f$, $x - a \in \mathcal{D}_f$ and $f(x + a) = f(x - a) = f(x)$.

Examples.

(1) Every $f : x \mapsto x^{2n}$ with $n \in \mathbb{N}$ is even on $\mathbb{R}$. Each $f : x \mapsto x^{2n+1}$ is odd.
(2) Functions sin and cos are periodic on $\mathbb{R}$, their period is 2π, but also 4π, 6π etc.
(3) tg is periodic on its domain, i.e. on $\mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi;\, k \in \mathbb{Z}\right\}$.
(4) Constant function $f : x \mapsto c$ with $c \in \mathbb{R}$ is periodic with any period $a \in \mathbb{R}$.

Remark. If for a periodic function the smallest period $a \in \mathbb{R}$ exists (which is the case of sin, cos with 2π and tg with π), then such an a is sometimes called *primitive period.*

Definition 3.8. Let f be a function, $M \subset \mathcal{D}_f$. We say that f is

- *bounded from above on* M, if
$$(\exists K \in \mathbb{R})\,(\forall x \in M)\, f(x) \leq K,$$
- *bounded from below on* M, if
$$(\exists K \in \mathbb{R})\,(\forall x \in M)\, f(x) \geq K,$$
- *bounded on* M, if
$$(\exists K > 0)\,(\forall x \in M)\, |f(x)| \leq K,$$
- *constant on* M, if $f(x) = f(y)$ for each x, $y \in M$.

Definition 3.9. Let $c \in \mathbb{R}$ and let $\varepsilon > 0$. We define

- $B_\varepsilon(c) = (c - \varepsilon, c + \varepsilon)$ *(open) neighborhood of* c,
- $P_\varepsilon(c) = B_\varepsilon(c) \setminus \{c\}$ *punctured neighborhood of* c,
- $P_\varepsilon(+\infty) = B_\varepsilon(+\infty) = (1/\varepsilon, +\infty)$ *neighborhood* and *punctured neighborhood of* $+\infty$,
- $P_\varepsilon(-\infty) = B_\varepsilon(-\infty) = (-\infty, -1/\varepsilon)$ *neighborhood* and *punctured neighborhood of* $-\infty$.

Definition 3.10. We say that $A \in \mathbb{R}^*$ is a *limit* of function f at the point $c \in \mathbb{R}^*$ if
$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall x \in P_\delta(c))\, f(x) \in B_\varepsilon(A)$$
and denote this fact by $\lim_{x \to c} f(x) = A$.

Remark. Notice that for δ in the previous definition, $P_\delta(c) \subset \mathcal{D}_f$, i.e. f is defined on some punctured neighbourhood of c.

Definition 3.11. Let $c \in \mathbb{R}$, $\varepsilon > 0$. We define

- $B_\varepsilon^+(c) = \langle c, c + \varepsilon)$ *right neighbourhood of* c,
- $B_\varepsilon^-(c) = (c - \varepsilon, c\rangle$ *left neighbourhood of* c,
- $P_\varepsilon^+(c) = (c, c + \varepsilon)$ *right punctured neighbourhood of* c,
- $P_\varepsilon^-(c) = (c - \varepsilon, c)$ *left punctured neighbourhood of* c.

Further, we put

- $B_\varepsilon^-(+\infty) = P_\varepsilon^-(+\infty) = B_\varepsilon(+\infty)$,
- $B_\varepsilon^+(-\infty) = P_\varepsilon^+(-\infty) = B_\varepsilon(-\infty)$.

Definition 3.12. Let $A \in \mathbb{R}^*$, $c \in \mathbb{R} \cup \{-\infty\}$. We say that A is *limit from the right* of a function f at c if
$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall x \in P_\delta^+(c))\, f(x) \in B_\varepsilon(A)$$
and denote it $\lim_{x \to c+} f(x) = A$. Similarly, *limit from the left*, $\lim_{x \to c-} f(x)$, defines for $c \in \mathbb{R} \cup \{+\infty\}$.

Example.

$$\lim_{x\to 0+}\frac{1}{x}=+\infty,\ \lim_{x\to 0-}\frac{1}{x}=-\infty,\ \lim_{x\to 0}\frac{1}{x}\text{ does not exist.}$$

Definition 3.13. We say that a function f is *continuous at* $c\in\mathbb{R}$ if $\lim_{x\to c}f(x)=f(c)$.

Definition 3.14. We say that a function f is *continuous at* $c\in\mathbb{R}$ *from the right (from the left, respectively)* if $\lim_{x\to c+}f(x)=f(c)$ ($\lim_{x\to c-}f(x)=f(c)$, resp.).

Example. $f(x)=\sqrt{x}:\langle 0,+\infty)\to\mathbb{R}$ is continuous at each $x\in\mathbb{R}^+$ (why?) and continuous from the right at $x=0$.

Theorem 3.1. *Let $c\in\mathbb{R}^*$. Each function has at most one limit at c.*

Proof. By contradiction: assume two different values a, b satisfy the definition of limit of f at c. If both $a,b\in\mathbb{R}$, put $\varepsilon=\frac{|b-a|}{3}$. Notice that $B_\varepsilon(a)\cap B_\varepsilon(b)=\emptyset$ then. If some of a, b is infinite, it is still easy to find $\varepsilon>0$ such that $B_\varepsilon(a)\cap B_\varepsilon(b)=\emptyset$. Since a is a limit of f at c, there exists $\delta_1>0$ such that

$$(\forall x\in P_{\delta_1}(c))\,f(x)\in B_\varepsilon(a).$$

And since b is a limit of f at c, there exists $\delta_2>0$ such that

$$(\forall x\in P_{\delta_2}(c))\,f(x)\in B_\varepsilon(b).$$

Let $\delta=\min\{\delta_1,\delta_2\}$ and take arbitrary $x\in P_\delta(c)$. For such x, $f(x)\in B_\varepsilon(a)\cap B_\varepsilon(b)$ – a contradiction. □

Theorem 3.2. *Suppose that a function f has a proper limit at $c\in\mathbb{R}^*$ (i.e., $\lim_{x\to c}f(x)\in\mathbb{R}$). Then there exists $\delta>0$ such that f is bounded on $P_\delta(c)$.*

Proof. Denote $\lim_{x\to c}f(x)=A\in\mathbb{R}$ and put $\varepsilon=1$. According to the definition of limit, there is $\delta>0$ such that

$$(\forall x\in P_\delta(c))\,f(x)\in B_\varepsilon(A)=(A-1,A+1).$$

Hence $A-1\in\mathbb{R}$ is a lower bound and $A+1\in\mathbb{R}$ is an upper bound of f on $P_\delta(c)$. □

Definition 3.15. Let $J\subset\mathbb{R}$ be an interval. We say that $f:J\to\mathbb{R}$ is *continuous on J if*

(1) *it is continuous at every interior point of J,*
(2) *it is continuous from the left at the right endpoint of J, if it belongs to J,*
(3) *it is continuous from the right at the left endpoint of J, if it belongs to J.*

Theorem 3.3 (Arithmetics of Limits). *Let $c\in\mathbb{R}^*$, let $\lim_{x\to c}f(x)=A\in\mathbb{R}^*$, $\lim_{x\to c}g(x)=B\in\mathbb{R}^*$. Then*

(1) $\lim_{x\to c}(f+g)(x)=A+B$, *if $A+B$ is defined,*
(2) $\lim_{x\to c}(f\cdot g)(x)=A\cdot B$, *if $A\cdot B$ is defined,*
(3) $\lim_{x\to c}\left(\frac{f}{g}\right)(x)=\frac{A}{B}$, *if $\frac{A}{B}$ is defined (in particular, if $B\neq 0$).*

Remark. An analogous theorem holds for $\lim_{x\to c+}$ and $\lim_{x\to c-}$.

Example. We shall see later that $\lim_{x\to 0}\frac{\sin x}{x}=1$. Let us apply this fact to compute $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

We rearrange the expression:

$$\frac{1-\cos x}{x^2}=\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x}=\frac{1-\cos^2 x}{x^2\cdot(1+\cos x)}=\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x}.$$

The first function has limit $1^2=1$ at 0 (we use arithmetics of limit), while the second one is continuous at 0, so it suffices to put $x=0$ there. Hence $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$.

Proposition 3.4. *Let $c\in\mathbb{R}^*$. Let $g>0$ on some $P_\delta(c)$ and $\lim_{x\to c} g(x)=0$. Further, let $\lim_{x\to c} f(x)=A>0$, $A\in\mathbb{R}^*$. Then $\lim_{x\to c}\frac{f(x)}{g(x)}=+\infty$.*

Theorem 3.5 (Limits and Inequalities). *Let $c\in\mathbb{R}$, let f, g, h be functions.*

(1) *Let $\lim_{x\to c} f(x)>\lim_{x\to c} g(x)$. Then there exists a punctured neighbourhood $P_\delta(c)$ such that $(\forall x\in P_\delta(c))\ f(x)>g(x)$.*
(2) *Let $f(x)\le g(x)$ on $P_\delta(c)$, let $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then $\lim_{x\to c} f(x)\le\lim_{x\to c} g(x)$.*
(3) *(Sandwich Theorem) Let $(\forall x\in P_\delta(c))\ f(x)\le h(x)\le g(x)$. Suppose that $\lim_{x\to c} f(x)=\lim_{x\to c} g(x)$. Then $\lim_{x\to c} h(x)$ exists and is equal to $lim_{x\to c}f(x)$.*

Remark. The same theorems hold for $\lim_{x\to c+}$ and $\lim_{x\to c-}$.

Theorem 3.6 (Limit of Composition). *Let c, D, $A\in\mathbb{R}^*$, $\lim_{x\to c} g(x)=D$, $\lim_{y\to D} f(y)=A$ and at least one of the following conditions is satisfied*

(1) $(\exists\eta>0)\ (\forall x\in P_\eta(c))\ g(x)\ne D$ *or*
(2) *f is continuous at D.*

Then $\lim_{x\to c} f(g(x))=A$.

Remark. The first conditions says that the *inner function* does not meet its limit on some punctured neighbourhood of c. The second condition expresses continuity of the *outer function* at the respective point.

Example. The *signum* function returns the sign of a given real number:

$$\operatorname{sgn} x=\begin{cases}-1 \text{ for } x<0,\\ 0 \text{ for } x=0,\\ 1 \text{ for } x>0.\end{cases}$$

It is bounded and monotone, but neither connected at 0 from the left nor from the right.

Example. If none of the conditions of Theorem 3.6 is satisfied, then the limit of the composed function need not be as expected. We consider $c=D=0$, $A=1$. Let

$$g(x)=\begin{cases}0 \text{ for } x\ne 0,\\ 1 \text{ for } x=0,\end{cases}$$

then $\lim_{x\to 0} g(x)=0$, but (1) fails – to the contrary, $g(x)\equiv 0$ on every $P_\eta(0)$. Let $f(x)=|\operatorname{sqn} x|$ – a function discontinuous at 0. Now, $\lim_{y\to 0} f(y)=1$, while $\lim_{x\to 0} f(g(x))=\lim_{x\to 0} f(0)=0$.

Theorem 3.7 (Heine). *Let A, $c \in \mathbb{R}^*$, let f be defined on $P_\delta(c)$. Then the following are equivalent:*

(1) $\lim_{x\to c} f(x) = A$,
(2) $(\forall x_n \in \mathcal{D}_f,\ x_n \neq c)\ \big(x_n \to c \Rightarrow f(x_n) \to A\big)$.

Example. The function $\sin x$ has no limit at $+\infty$: consider sequences $\{a_n\}$ and $\{b_n\}$ with $a_n = n\pi$ and $b_n = \pi/2 + 2n\pi$ ($n \in \mathbb{N}$). Then $a_n \to +\infty$ and $b_n \to +\infty$, $\sin a_n \to 0$ and $\sin b_n \to 1$. If there was a limit $\lim_{x\to+\infty} \sin x$, then the two sequences $\{\sin a_n\}$ and $\{\sin b_n\}$ would have the same limit equal to $\lim_{x\to+\infty} \sin x$.

We can derive that there is no limit of $\sin \frac{1}{x}$ in 0 either.

Example on limit of composition. Let f be continuous at 0. Then

$$\lim_{x\to+\infty} f(1/x) = f(0).$$

How Theorem 3.6 applies here? We find $c = +\infty$, $D = 0 = \lim_{x\to+\infty} \frac{1}{x}$, $A = f(0)$. The outer function f is continuous at D as well as the inner function $\frac{1}{x}$ does not reach its limit 0 on any $P_\delta(+\infty)$.

Remark. On computation of limits of composed functions, conditions of Theorem 3.6 must always be verified.

Example. We shall compute

$$\lim_{x\to\frac{\pi}{6}} \frac{\sin(x - \frac{\pi}{6})}{x - \frac{\pi}{6}}$$

using the 'tabular' limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Here, the outer function, $f(y) = \frac{\sin y}{y}$ is *not* continuous at 0! Further, $c = \frac{\pi}{6}$, $g(x) = x - \frac{\pi}{6}$, $D = \lim_{x\to\frac{\pi}{6}} g(x) = 0$, $A = \lim_{x\to 0} \frac{\sin x}{x} = 1$. We have to verify the second condition. Indeed, $g(x) \neq 0$ outside $\frac{\pi}{6}$, i.e. on any $P_\delta(\frac{\pi}{6})$, because g is one-to-one.

Theorem 3.8 (Limit of monotone function). *Let f be monotone on (a,b), a, $b \in \mathbb{R}^*$. Then there exists $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$.*

In the sequel, we shall deal with functions continuous on intervals.

Theorem 3.9 (Bolzano, Darboux). *Let f be a continuous function on $\langle a,b\rangle$, a, $b \in \mathbb{R}$, $f(a) < f(b)$. Then for every $c \in (f(a), f(b))$ there exists $\xi \in (a,b)$ such that $f(\xi) = c$.*

Theorem 3.10. *Let J be an interval, let $f : J \to \mathbb{R}$ be continuous on J. Then $f[J]$ is an interval or a one-point set.*

The key is to prove the following lemma.

Lemma 3.11. *Let $\emptyset \neq C \subset \mathbb{R}$ be convex, i.e. a, $b \in C$, $a < c < b$ imply $c \in C$. Then C is an interval or a one-point set.*

Theorem 3.12. *Let f be a continuous function on an interval $\langle a, b\rangle$. Then f is bounded on $\langle a, b\rangle$.*

The proof (by contradiction) uses Bolzano-Weierstrass Theorem.

Definition 3.16. Let $M \subset \mathbb{R}$, $x \in M$ and let $M \subset \mathcal{D}_f$ for a function f. We say that f attains at x its

- *maximum on M* (and denote $f(x) = \max_M f$) if $(\forall y \in M)\, f(y) \leq f(x)$,
- *minimum on M* (and denote $f(x) = \min_M f$) if $(\forall y \in M)\, f(y) \geq f(x)$.

The point x is called *point of maximum of f* (*point of minimum of f*, respectively).

Definition 3.17 = 3.16'. Let $M \subset \mathbb{R}$, $x \in M$ and let $M \subset \mathcal{D}_f$ for a function f. We say that f attains at x

- *local maximum with respect to M* if there exists $\delta > 0$ such that $(\forall y \in P_\delta(x) \cap M)\, f(y) \leq f(x)$,
- *local minimum with respect to M* if there exists $\delta > 0$ such that $(\forall y \in P_\delta(x) \cap M)\, f(y) \geq f(x)$.

Examples.

(1) $\sin x$ attains (local) maximum at every $\frac{\pi}{2} + 2k\pi$ $(k \in \mathbb{Z})$, (local) minimum at $-\frac{\pi}{2} + 2k\pi$ $(k \in \mathbb{Z})$.
(2) A typical local maximum which is not maximum is attained at 0 by the function $\big||x| - 1\big|$ (draw the graph!).

Theorem 3.13. *Let f be continuous on $\langle a, b\rangle$. Then f attains its maximum and minimum on $\langle a, b\rangle$.*

Remark. Notice that the requirement on the interval being closed is essential: there are functions that map a bounded open interval onto the whole $\mathbb{R}$.

Theorem 3.14. *Let f be an increasing continuous function on an interval J. Then f^{-1} is continuous and increasing on $f[J]$.*

Elementary functions.

Theorem 3.15 + Definition. *There exists a unique function logarithm (*$\log$*) with the following properties:*

(L1) $\mathcal{D}_{\log} = \mathbb{R}^+ = (0, +\infty)$ *and* $\log$ *is increasing on* $(0, +\infty)$,
(L2) $\big(\forall x, y \in (0, +\infty)\big)\ \log(x \cdot y) = \log x + \log y$,
(L3) $\lim_{x \to 1} \frac{\log x}{x-1} = 1$.

Definition 3.17. *Exponential function* ($x \mapsto \exp(x)$ or e^x) is defined as the the inverse function to log.

Definition 3.18. Let a, $b \in \mathbb{R}$, $a > 0$. The number a^b is defined as $a^b = \exp(b \cdot \log a)$.

Remarks.

(1) Is the last definition correct? I.e., is $a^n = \exp(n \cdot \log a)$, $a^{-n} = \exp(-n \cdot \log a)$, $a^{\frac{1}{n}} = \exp\left(\frac{1}{n} \cdot \log a\right)$ for every $n \in \mathbb{N}$?

(2) What are $\mathcal{D}_{\exp}$ and further properties of the function? To answer (1) and (2), we have to prove more about log.

Proposition 3.16 (Further properties of $\log$**).**

(1) $\log 1 = 0$,
(2) $(\forall x \in (0,+\infty))\ \log\left(\frac{1}{x}\right) = -\log x$,
(3) $(\forall x \in (0,+\infty))\ (\forall n \in \mathbb{Z})\ \log(x^n) = n \cdot \log x$,
(4) $\lim_{x\to+\infty} \log x = +\infty$, $\lim_{x\to 0+} \log x = -\infty$,
(5) $\log$ *is continuous on* $(0,+\infty)$,
(6) $\mathcal{R}_{\log} = \mathbb{R}$.

Proposition 3.17 (Properties of $\exp$**).**

(1) $\mathcal{D}_{\exp} = \mathbb{R}$, $\mathcal{R}_{\exp} = (0,+\infty)$,
(2) $\exp$ *is increasing on* $\mathbb{R}$,
(3) $\exp$ *is continuous on* $\mathbb{R}$, $\lim_{x\to+\infty} \exp x = +\infty$, $\lim_{x\to-\infty} \exp x = 0$,
(4) $\exp 0 = 1$,
(5) $(\forall x, y \in \mathbb{R})\ \exp(x+y) = \exp x \cdot \exp y$,
(6) $\lim_{x\to 0} \frac{\exp x - 1}{x} = 1$.

Definition 3.19. e is the unique number such that $\log e = 1$.

Theorem 3.18. *The number* e *is irrational,* $e \doteq 2.71828$, $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$.

Definition 3.20. Let $a > 0$, $a \neq 1$. *Logarithm of* x *to the base* a defines as $\log_a x = \frac{\log x}{\log a}$ for every $x \in (0,+\infty)$.

Theorem 3.19 + Definition. *There exists a unique* $\pi > 0$, $\pi \in \mathbb{R}$ *and a unique function sine* ($\sin$) *such that*

(S1) $\mathcal{D}_{\sin} = \mathbb{R}$,
(S2) $\sin$ *is increasing on* $\langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$,
(S3) $\sin 0 = 0$,
(S4) $(\forall x, y \in \mathbb{R})\ \sin(x+y) = \sin x \cdot \sin\left(\frac{\pi}{2} - y\right) + \sin\left(\frac{\pi}{2} - x\right) \cdot \sin y$,
(S5) $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Proposition 3.20 (Further properties of $\sin$**).**

(1) $\sin \frac{\pi}{2} = 1$,
(2) $(\forall x \in \mathbb{R})\ |\sin x| \leq 1$,
(3) $\sin$ *is continuous on* $\mathbb{R}$,
(4) $\sin$ *is odd and* 2π*-periodic.*

Definition 3.21 (Further trigonometric functions). The following functions are defined:

- *cosine:* $\cos x = \sin\left(\frac{\pi}{2} - x\right)$, $x \in \mathbb{R}$,
- *tangent:* $\operatorname{tg} x = \frac{\sin x}{\cos x}$, $x \in \mathbb{R} \setminus \{(2k+1)\frac{\pi}{2};\ k \in \mathbb{Z}\}$,
- *cotangent:* $\operatorname{cotg} x = \frac{\cos x}{\sin x}$, $x \in \mathbb{R} \setminus \{k\pi;\ k \in \mathbb{Z}\}$.

Definition 3.22 (Cyclometric functions). The symbol $\restriction$ stands for restriction of a particular function to a given domain.

- *arcsine:* $\arcsin = \left(\sin \restriction \langle -\pi/2, \pi/2\rangle\right)^{-1}$,
- *arccosine:* $\arccos = \left(\cos \restriction \langle 0, \pi\rangle\right)^{-1}$,
- *arctangent:* $\operatorname{arctg} = \left(\operatorname{tg} \restriction (-\pi/2, \pi/2)\right)^{-1}$,
- *arccotangent:* $\operatorname{arccotg} = \left(\operatorname{cotg} \restriction (0, \pi)\right)^{-1}$.

Remark. It is easy to see that $\mathcal{D}_{\text{arcsin}} = \mathcal{D}_{\text{arccos}} = \langle -1, 1\rangle$ and $\mathcal{D}_{\text{arctg}} = \mathcal{D}_{\text{arccotg}} = \mathbb{R}$.

Proposition 3.21. *All the functions defined in 3.21–3.22 are continuous on their domains.*

Derivative.

Definition 3.23. Let f be a real function, $a \in \mathbb{R}$. *Derivative of f at a* is defined by the formula

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

if the limit on the right exists.

Remark. Existence of derivative in a involves the fact that f is defined not only in a but on some neighbourhood $B_\delta(a)$.

Definition 3.24. Let f be a real function, $a \in \mathbb{R}$. *Derivative of f at a from the right* (*from the left,* respectively) is defined by the formula

$$f'_+(a) = \lim_{h\to 0+} \frac{f(a+h) - f(a)}{h} \quad \left(f'_-(a) = \lim_{h\to 0-} \frac{f(a+h) - f(a)}{h}, \text{ resp.}\right),$$

if the limits on the right exist.

Examples.

(1) Let us compute $f'(x)$ for $f(x) = x^n$ with $n \in \mathbb{N}$ at given (but arbitrary) $x \in \mathbb{R}$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{\big((x+h) - x\big) \cdot \big((x+h)^{n-1} + (x+h)^{n-2} \cdot x + \cdots + x^{n-1}\big)}{h} \\ &= \lim_{h\to 0} \big((x+h)^{n-1} + (x+h)^{n-2} \cdot x + \cdots + x^{n-1}\big) \\ &= n \cdot x^{n-1}. \end{aligned}$$

(2) For $f(x) = \operatorname{sgn} x$ (cf. Example after Theorem 3.6), $f'(0)$ can be computed directly as $+\infty$. Notice that, then, sgn is a function discontinuous in 0, but has a derivative there. The following theorem precises the relation between continuity and existence of derivative.

Theorem 3.22. *If a function f has a proper derivative in a point a (i.e. $f'(a) \in \mathbb{R}$), then f is continuous in a.*

Remarks.

(1) Alternatively, derivative of f at a can be defined as $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$.
(2) $f'(a)$ either exists and

$$\begin{cases} \text{is proper, i.e. } f'(a) \in \mathbb{R}, \\ \text{is improper, i.e. } f'(a) \in \{+\infty, -\infty\}, \end{cases}$$

or does not exist.
(3) Geometric meaning of derivative $f'(a)$ is the slope of the tangent line of the graph of f in the point a.

Theorem 3.23 (Arithmetics of Derivatives). *Let f, g have proper derivatives at $a \in \mathbb{R}$. Then*

(i) $(f+g)'(a) = f'(a) + g'(a)$, $(\alpha f)'(a) = \alpha \cdot f'(a)$ *for every* $\alpha \in \mathbb{R}$,
(ii) $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$,
(iii) *if* $g(a) \neq 0$ *then* $(f/g)'(a) = \frac{f'(a)\cdot g(a) - f(a)\cdot g'(a)}{g^2(a)}$.

Derivatives of some elementary functions.

- $\log' x = \frac{1}{x}$, $x \in (0, +\infty)$
- $(e^x)' = e^x$, $x \in \mathbb{R}$
- $\sin' x = \cos x$, $x \in \mathbb{R}$
- $\cos' x = \sin x$, $x \in \mathbb{R}$
- $\mathrm{tg}' x = \frac{1}{\cos^2 x}$, $x \in \bigcup_{k\in\mathbb{Z}} \left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right)$
- $\mathrm{cotg}' x = -\frac{1}{\sin^2 x}$, $x \in \bigcup_{k\in\mathbb{Z}} (k\pi, (k+1)\pi)$

Theorem 3.24 (Derivative of Composed Function). *Let x_0, $y_0 \in \mathbb{R}$, $g(x_0) = y_0$, $g'(x_0) \in \mathbb{R}$, $f'(y_0) \in \mathbb{R}$. Then $(f \circ g)'(x_0) = f'(y_0) \cdot g'(x_0)$.*

- We have proved that $(x^n)' = n \cdot x^{n-1}$, $x \in \mathbb{R}$, for every $n \in \mathbb{N}$ (in particular, $const' = 0$). Let us extend this formula to arbitrary exponent $\alpha \in \mathbb{R}$. Notice that we apply Theorem 3.24.

$$\begin{aligned} (x^\alpha)' &= \big(\exp(\alpha \cdot \log x)\big)' = \exp'(\alpha \cdot \log x) \cdot \alpha \cdot \log' x \\ &= \exp(\alpha \cdot \log x) \cdot \alpha \cdot \frac{1}{x} = x^\alpha \cdot \alpha \cdot \frac{1}{x} = \alpha \cdot x^{\alpha-1} \end{aligned}$$

for every $x \in (0, +\infty)$.

Theorem 3.25 (Derivative of Inverse Function). *Let f be continuous and increasing (decreasing, respectively) on an interval (a, b). Let f have a proper nonzero $f'(x_0)$ at $x_0 \in (a, b)$. Then f^{-1} has $(f^{-1})'$ at $y_0 = f(x_0)$ and*

$$(f^{-1})'(y_0) = \frac{1}{f'\big(f^{-1}(y_0)\big)}.$$

Theorem 3.25 applies in computing derivatives of cyclometric functions.

- $\arcsin' x = \frac{1}{\sqrt{1-x^2}}$, $x \in (-1,1)$
- $\arccos' x = -\frac{1}{\sqrt{1-x^2}}$, $x \in (-1,1)$
- $\operatorname{arctg}' x = \frac{1}{1+x^2}$, $x \in \mathbb{R}$
- $\operatorname{arccotg}' x = -\frac{1}{1+x^2}$, $x \in \mathbb{R}$

Derivative and its relation to local extrema.

Theorem 3.26 (Necessary Condition for Local Extremum). *Let x_0 be a point of local extremum of f. Then either $f'(x_0)$ does not exist or $f'(x_0) = 0$.*

Remark. Let $f : \langle a, b\rangle \to \mathbb{R}$. The function can attain its maxima/minima on $\langle a, b\rangle$ at

(1) points a, b,
(2) points $x_0 \in (a, b)$ such that $f'(x_0)$ does not exist,
(3) points $x_0 \in (a, b)$ such that $f'(x_0) = 0$ (Theorem 3.26).

Notice that contnuity of a function defined on **a closed bounded interval** $\langle a, b\rangle$ guarantees existence of points of maxima and minima on $\langle a, b\rangle$ (Theorem 3.13).

Deeper theorems on derivatives.

Theorem 3.27 (Rolle). *Let a, $b \in \mathbb{R}$, $a < b$. Let a function f satisfy*

(i) *f is continuous on $\langle a, b\rangle$,*
(ii) *f has (proper or improper) derivative at every point of (a, b),*
(iii) *$f(a) = f(b)$.*

Then there is $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Theorem 3.28 (Lagrange). *Let a, $b \in \mathbb{R}$, $a < b$, let f be continuous on $\langle a, b\rangle$ and have (proper or improper) derivative on (a, b). Then there is $\xi \in (a, b)$ such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Remarks.

(1) There can be more than one point ξ in Theorems 3.27, 3.28.
(2) Theorems 3.27, 3.28 are also referred to as Mean Value Theorems.

Definition 3.25. For interval J with endpoints a, $b \in \mathbb{R}^*$, $a < b$, we denote $\operatorname{int} J$ its *interior*, i.e. $\operatorname{int} J = (a, b)$.

Theorem 3.29 (Monotonicity and the Sign of the Derivative). *Let $J \subset \mathbb{R}$ be an interval, f continuous on J and let f' exist at each point of $\operatorname{int} J$. Then*

(1) *if $f'(x) > 0$ for each $x \in \operatorname{int} J$ then f is increasing on J,*
(2) *if $f'(x) < 0$ for each $x \in \operatorname{int} J$ then f is decreasing on J,*
(3) *if $f'(x) \geq 0$ for each $x \in \operatorname{int} J$ then f is nondecreasing on J,*
(4) *if $f'(x) \leq 0$ for each $x \in \operatorname{int} J$ then f is nonincreasing on J.*

Theorem 3.30 (l'Hospital Rule). *Let f, g have proper derivatives f', g' on some $P_\delta(a)$, $a \in \mathbb{R}^*$, and suppose $\lim_{x\to a} \frac{f'}{g'}$ exists. If, moreover,*

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ *or*
(ii) $\lim_{x\to a} |g(x)| = +\infty$,

then $\lim_{x\to a} \frac{f}{g}$ exists and is equal to $\lim_{x\to a} \frac{f'}{g'}$.

Remarks.

(1) Conditions (i), (ii) of Theorem 3.30 must be verified before use of l'Hospital Rule. Always write that the limit is, e.g., 'of the type $\frac{0}{0}$' or 'of the type $\frac{\infty}{\infty}$'. Generally, the rule does not hold for other values of $\lim_{x\to a} f$, $\lim_{x\to a} g$ – try to find a counterexample.
(2) The Rule can be used repeatedly, e.g.

$$\lim_{x\to+\infty} \frac{e^x}{x^3} = \lim_{x\to+\infty} \frac{e^x}{3x^2} = \lim_{x\to+\infty} \frac{e^x}{6x} = \lim_{x\to+\infty} \frac{e^x}{6} = +\infty.$$

We apply l'Hospital Rule for 'limit of the type $\frac{\infty}{\infty}$' three times here.
(3) Computing with derivatives f', g' instead of f, g does not always ease the situation, e.g., if derivative of product or composed functions occurs in the numerator and/or denominator of the expression.

Theorem 3.31. *Let f be continuous from the right at $a \in \mathbb{R}$ and let $\lim_{x\to a+} f'(x)$ exist. Then $f'_+(a)$ exists and*

$$f'_+(a) = \lim_{x\to a+} f'(x).$$

Similarly from the left.

Example. The function arcsin is continuous on $\langle -1, 1\rangle$, in particular, continuous from the right at -1 and from the left at 1.

$$\lim_{x\to-1+} \arcsin' x = \lim_{x\to-1+} \frac{1}{\sqrt{1-x^2}} = +\infty = \lim_{x\to 1-} \arcsin' x,$$

hence $\arcsin'_+(-1) = \arcsin'_-(1) = +\infty$.

Definition 3.26. Let $n \in \mathbb{N}$, $a \in \mathbb{R}$, let f have proper n-th derivative $f^{(n)}$ on a neighbourhood of a. Then $(n+1)$*-st derivative of f at a* is defined by

$$f^{(n+1)}(a) = \left(f^{(n)}\right)'(a) = \lim_{h\to 0} \frac{f^{(n)}(a+h) - f^{(n)}(a)}{h}.$$

Formally, we put $f^{(0)} = f$. Small order derivatives have duplicit notation: $f' = f^{(1)}$, $f'' = f^{(2)}$, $f''' = f^{(3)}$.

Convex and Concave Functions.

Definition 3.27. Let f have a proper first derivative at $a \in \mathbb{R}$. We call the set

$$T_a = \big\{[x, y] \in \mathbb{R};\ y = f(a) + f'(a) \cdot (x - a)\big\}$$

tangent line to the graph of f at $[a, f(a)]$.

We say that $[x, f(x)]$ is

- *below the tangent line* T_a if $f(x) < f(a) + f(a) + f'(a) \cdot (x - a)$,
- *above the tangent line* T_a if $f(x) > f(a) + f(a) + f'(a) \cdot (x - a)$.

Definition 3.28. Let $f'(a) \in \mathbb{R}$. We say that a is an *inflection point* of f if there exists a $\Delta > 0$ such that

(i) $\big(\forall x \in (a - \Delta, a)\big)$ $[x, f(x)]$ is below T_a and
(ii) $\big(\forall x \in (a, a + \Delta)\big)$ $[x, f(x)]$ is above T_a

or

(i) $\big(\forall x \in (a - \Delta, a)\big)$ $[x, f(x)]$ is above T_a and
(ii) $\big(\forall x \in (a, a + \Delta)\big)$ $[x, f(x)]$ is below T_a.

Theorem 3.32. *Let $a \in \mathbb{R}$ be an inflection point of f. Then $f''(a)$ does not exist or is equal to 0.*

Remarks.

(1) (Analogy to search for extrema.) Let f have proper derivative everywhere on (a, b). Then inflection points of f on (a, b) are points c at which either $f''(c)$ does not exist or $f''(c) = 0$.
(2) $f''(c) = 0$ does not imply c is an inflection point of f – consider, e.g., $f(x) = x^4$, $c = 0$. Here $f''(c) = 0$ but all the graph is above the tangent line which is the x-axis in this case.

Theorem 3.33. *Let f have a continuous derivative on (a, b) and $x_0 \in (a, b)$. Suppose that $(\forall x \in (a, x_0))$ $f''(x) > 0$ and $(\forall x \in (x_0, b))$ $f''(x) < 0$. Then x_0 is an inflection point of f.*

Definition 3.29. Let I be an interval. We say that f is

- *convex* on I if $(\forall x_1, x_2 \in I)$ $(\forall \lambda \in \langle 0, 1\rangle)$ $f\big(\lambda x_1 + (1 - \lambda)x_2\big) \le \lambda f(x_1) + (1 - \lambda)f(x_2)$
- *concave* on I if $(\forall x_1, x_2 \in I)$ $(\forall \lambda \in \langle 0, 1\rangle)$ $f\big(\lambda x_1 + (1 - \lambda)x_2\big) \ge \lambda f(x_1) + (1 - \lambda)f(x_2)$
- *strictly convex* on I if $(\forall x_1, x_2 \in I,\ x_1 \neq x_2)$ $\big(\forall \lambda \in (0, 1)\big)$ $f\big(\lambda x_1 + (1 - \lambda)x_2\big) < \lambda f(x_1) + (1 - \lambda)f(x_2)$
- *strictly concave* on I if $(\forall x_1, x_2 \in I,\ x_1 \neq x_2)$ $\big(\forall \lambda \in (0, 1)\big)$ $f\big(\lambda x_1 + (1 - \lambda)x_2\big) > \lambda f(x_1) + (1 - \lambda)f(x_2)$

Remark. $\lambda x_1 + (1 - \lambda)x_2$ with $\lambda \in \langle 0, 1\rangle$ expresses a typical element of the segment connecting points x_1 and x_2.

Examples. Natural logarithm log is strictly concave on $(0, +\infty)$, exp is strictly convex on $\mathbb{R}$. This is obvious from the shapes of the graphs but computation and estimates using definition might be difficult. The following theorem gives an easy criterion of convexity/concaveness for a class of functions.

Theorem 3.34 (Second Derivative and Convexity). *Let f have a proper second derivative f'' on (a, b), $a < b$.*

(1) *If $f''(x) > 0$ for every $x \in (a, b)$ then f is strictly convex on (a, b).*
(2) *If $f''(x) < 0$ for every $x \in (a, b)$ then f is strictly concave on (a, b).*
(3) *If $f''(x) \geq 0$ for every $x \in (a, b)$ then f is convex on (a, b).*
(4) *If $f''(x) \leq 0$ for every $x \in (a, b)$ then f is concave on (a, b).*

Example. For $f(x) = \log x$ is $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$ which is negative on all the domain of log. It follows that log is indeed strictly concave on $(0, +\infty)$.

Definition 3.30. We say that a function $x \mapsto ax + b$, a, $b \in \mathbb{R}$, is *asymptote of f at $+\infty$* (*at $-\infty$*, resp.) if

$$\lim_{x \to +\infty} \big(f(x) - ax - b\big) = 0 \ \Big(\lim_{x \to -\infty} \big(f(x) - ax - b\big) = 0, \text{ resp.}\Big).$$

Theorem 3.35. *A function f has asymptote $x \mapsto ax + b$ in $+\infty$ if and only if*

$$\lim_{x \to +\infty} \frac{f(x)}{x} = a \in \mathbb{R} \text{ and } \lim_{x \to +\infty} \big(f(x) - ax\big) = b \in \mathbb{R}.$$

Remarks.

(1) Analogous theorem holds for $x \to -\infty$.
(2) Theorem 3.35 describes the way to compute parameters a, b of an asymptote (or to show that a function has no asymptote).

Investigation of a function f.

(1) Determine the domain $\mathcal{D}_f$ and the set of all points of continuity of f.
(2) Find out if the function is odd, even or periodic.
(3) Compute limits at all endpoints of $\mathcal{D}_f$ (if $\mathcal{D}_f$ is a union of intervals there may be more than two limits to investigate).
(4) Compute the first derivative f' in all points in which it exists, including derivatives from the right/left in $x \in \mathcal{D}_f$ in which $f'(x)$ does not exist. Use f' to find intervals of monotonicity of f, its local and global maxima/minima and the range $\mathcal{R}_f$.
(5) Compute the second derivative f'' in all points in which it exists. Use it to find intervals of convexity/concaveness of f and inflection points.
(6) Find asymptotes at $\pm\infty$ if they exist.
(7) Draw the graph of f. It may involve further computation, e.g. $f(x)$ at important points, $f'(x)$ at inflection points etc.