Problem 1 - version 1 Compute the inverse of the matrix $\mathbb{A}$. (Hint: Let $\mathbb{B}$ be the matrix made from $\mathbb{A}$ by multiplying the second row by $\frac{1}{3}$, the third row by $\frac{1}{9}$ and the fourth row by $\frac{1}{27}$. Compute first $\mathbb{B}^{-1}$ and then deduce the value of $\mathbb{A}^{-1}$.)

$$\mathbb{A} = \begin{pmatrix} 1 & 3 & 9 & 27 \\ 3 & 3 & 9 & 27 \\ 9 & 9 & 9 & 27 \\ 27 & 27 & 27 & 27 \end{pmatrix}$$

Sketch of the solution.
The matrix $\mathbb{B}$ equals

$$\mathbb{B} = \begin{pmatrix} 1 & 3 & 9 & 27 \\ 1 & 1 & 3 & 9 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

To compute the inverse we take the matrix

$$(\mathbb{B}|\mathbb{I}) = \left(\begin{array}{cccc|cccc} 1 & 3 & 9 & 27 & 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 9 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 3 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Using standard elementary row transformations we arrive to

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & -2 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right),$$

thus

$$\mathbb{B}^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} & 0 & 0 \\ \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & \frac{1}{2} & -2 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Finally, since $\mathbb{A}$ can be made from $\mathbb{B}$ by multiplying the second row by 3, the third row by 9 and the fourth row by 27, the matrix $\mathbb{A}^{-1}$ will be made from $\mathbb{B}^{-1}$ by multiplying the second column by $\frac{1}{3}$, the third column by $\frac{1}{9}$ and the fourth column by $\frac{1}{27}$. Therefore

$$\mathbb{A}^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{2}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & -\frac{2}{9} & \frac{1}{18} \\ 0 & 0 & \frac{1}{18} & -\frac{1}{54} \end{pmatrix}.$$

Approximate evaluation:
Correct beginning of computation of the inverse: 1 point.
Correct application of elementary row transformation: 5 points. (The computation should be, of course, given explicitly step by step, in order the way to the result can checked.)
Pointing out the value of $\mathbb{B}^{-1}$: 1 point.
Determining the value of $\mathbb{A}^{-1}$ including explanation: 3 points.

Problem 2 - version 2 Compute determinants of matrices $\mathbb{A}$ and $\mathbb{B}^T\mathbb{A}$, where $\mathbb{A}$ is given below and $\mathbb{B}$ is made from $\mathbb{A}$ by multiplying the first column by 17 and fifth column by $\frac{1}{5}$.

$$\mathbb{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 12 & 13 & 14 & 15 \\ 3 & 13 & 23 & 24 & 25 \\ 4 & 14 & 24 & 34 & 35 \\ 5 & 15 & 25 & 35 & 50 \end{pmatrix}$$

Sketch of solution: Standard rules for transformation and determinant yield:

$$\det \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 12 & 13 & 14 & 15 \\ 3 & 13 & 23 & 24 & 25 \\ 4 & 14 & 24 & 34 & 35 \\ 5 & 15 & 25 & 35 & 50 \end{pmatrix} = 25 \det \begin{pmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 12 & 13 & 14 & 3 \\ 3 & 13 & 23 & 24 & 5 \\ 4 & 14 & 24 & 34 & 7 \\ 1 & 3 & 5 & 7 & 2 \end{pmatrix} = 25 \det \begin{pmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 8 & 7 & 6 & 1 \\ 0 & 7 & 14 & 12 & 2 \\ 0 & 6 & 12 & 18 & 3 \\ 0 & 1 & 2 & 3 & 1 \end{pmatrix}$$

$$= 25 \det \begin{pmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & -9 & -18 & -7 \\ 0 & 0 & 0 & -9 & -5 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 1 & 2 & 3 & 1 \end{pmatrix} = 25 \det \begin{pmatrix} 0 & -9 & -18 & -7 \\ 0 & 0 & -9 & -5 \\ 0 & 0 & 0 & -3 \\ 1 & 2 & 3 & 1 \end{pmatrix}$$

$$= 25 \cdot 1 \cdot (-1)^{1+4} \det \begin{pmatrix} -9 & -18 & -7 \\ 0 & -9 & -5 \\ 0 & 0 & -3 \end{pmatrix}$$

$$= -25 \cdot (-9) \cdot (-9) \cdot (-3) = 25 \cdot 3^5.$$

Hence $\det \mathbb{A} = 25 \cdot 3^5$.
Further, $\det \mathbb{B}^T = \det \mathbb{B} = 17 \cdot \frac{1}{5} \cdot \det \mathbb{A} = 5 \cdot 17 \cdot 3^5$.
Finally: $\det(\mathbb{B}^T\mathbb{A}) = \det \mathbb{B}^T \cdot \det \mathbb{A} = 17 \cdot 5^3 \cdot 3^{10}$.

Approximate evaluation:
Computing $\det \mathbb{A}$: 7 points.
Deducing the value of $\det(\mathbb{B}^T\mathbb{A})$: 3 points.

Problem 1 - version 3

Find all the solutions of the system $\mathbb{A}\mathbf{x} = \mathbf{b}$ for the below given matrix $\mathbb{A}$ and given three right-hand side vectors $\mathbf{b}_1$, $\mathbf{b}_2$ a $\mathbf{b}_3$.

$$\mathbb{A} = \begin{pmatrix} 1 & 3 & 9 & 27 \\ 3 & 1 & 27 & 9 \\ 1 & 3 & 27 & 9 \\ 3 & 1 & 9 & 27 \end{pmatrix}, \mathbf{b}_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 2 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

Sketch of the solution:

We will use Gauss elimination method. The best possibility is to make the computation simultaneously for all the three vectors. I.e., take the augmented matrix

$$(\mathbb{A}|\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3) = \left(\begin{array}{cccc|ccc} 1 & 3 & 9 & 27 & 1 & 1 & 1 \\ 3 & 1 & 27 & 9 & 3 & 1 & 2 \\ 1 & 3 & 27 & 9 & 2 & 1 & 3 \\ 3 & 1 & 9 & 27 & 2 & 1 & 4 \end{array}\right)$$

and perform the elimination (i.e., transform the matrix to the row echelon form using elementary row transformation). We will get

$$\left(\begin{array}{cccc|ccc} 1 & 3 & 9 & 27 & 1 & 1 & 1 \\ 0 & -8 & 0 & -72 & 0 & -2 & -1 \\ 0 & 0 & 18 & -18 & 1 & 0 & 2 \\ 0 & -8 & -18 & -54 & -1 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|ccc} 1 & 3 & 9 & 27 & 1 & 1 & 1 \\ 0 & -8 & 0 & -72 & 0 & -2 & -1 \\ 0 & 0 & 18 & -18 & 1 & 0 & 2 \\ 0 & 0 & -18 & 18 & -1 & 0 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|ccc} 1 & 3 & 9 & 27 & 1 & 1 & 1 \\ 0 & -8 & 0 & -72 & 0 & -2 & -1 \\ 0 & 0 & 18 & -18 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{array}\right)$$

Now we can write down the solutions:

For $\mathbf{b}_1$: $[-9t + \frac{1}{2}, -9t, t + \frac{1}{18}, t]$, $t \in \mathbf{R}$.

For $\mathbf{b}_2$: $[-9t + \frac{1}{4}, -9t + \frac{1}{4}, t, t]$, $t \in \mathbf{R}$.

For $\mathbf{b}_3$: no solution.

Approximate evaluation:

Correct beginning of the elimination: 1 point.

The elimination itself: 4 points.

Deducing the formulas for solutions for $\mathbf{b}_1$ 2 points, for $\mathbf{b}_2$ 2 points.

Nonexistence of solution for $\mathbf{b}_3$: 1 point.

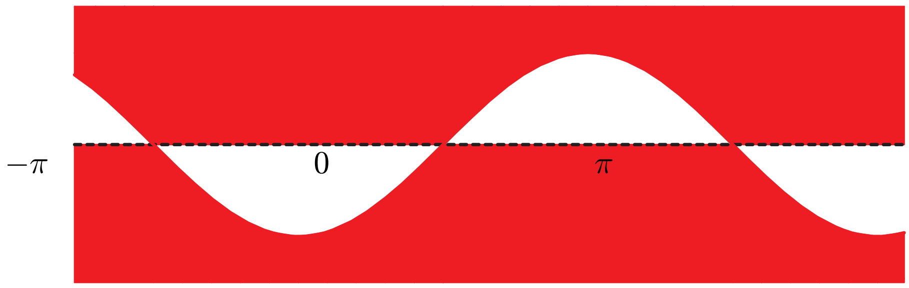
Problem 2 Determine and draw the domain of the function

$$f(x,y)=\sqrt{\frac{y-\sin x}{y}},$$

compute its partial derivatives with respect to all the variables at all points where they exist.
Sketch of the solution: Domain is the following:

$$D_f=\{[x,y]\in\mathbf{R}^2:(y>0 \text{ and } y\geq\sin x) \text{ or } (y<0 \text{ and } y\leq\sin x)\}.$$

Picture of the domain:



Partial derivatives are computed using standard rules:

$$\frac{\partial f}{\partial x}(x,y)=\frac{1}{2}\sqrt{\frac{y}{y-\sin x}}\cdot\frac{-\cos x}{y}$$
$$\frac{\partial f}{\partial y}(x,y)=\frac{1}{2}\sqrt{\frac{y}{y-\sin x}}\cdot\frac{\sin x}{y^2}.$$

Both formulas are valid at points $[x,y]$ satisfying $y>0$ and $y>\sin x$ or $y<0$ and $y<\sin x$.
It remains to check the points $[x,y]$ satisfying $y=\sin x$, $y\neq 0$.
At these points partial derivative with respect to y is not defined, since there is no vertical segment centered at the respective point contained in the domain.
Further, partial derivative with respect to x has a sense only in points $[\frac{\pi}{2}+2k\pi,1]$ and $[-\frac{\pi}{2}+2k\pi,-1]$ for $k\in\mathbf{Z}$, since otherwise there is no horizontal segment centered at the respective point contained in the domain.
Finally, $\frac{\partial f}{\partial x}(\frac{\pi}{2}+2k\pi,1)$ and $\frac{\partial f}{\partial x}(-\frac{\pi}{2}+2k\pi,-1)$ for $k\in\mathbf{Z}$ do not exist. This can be checked, for example, using the definition of partial derivatives.

$$\frac{\partial f}{\partial x}(\frac{\pi}{2}+2k\pi,1)=\lim_{x\to\frac{\pi}{2}+2k\pi}\frac{\sqrt{\frac{1-\sin x}{1}}-0}{x-(\frac{\pi}{2}+2k\pi)}=\lim_{x\to\frac{\pi}{2}}\frac{\sqrt{1-\sin x}}{x-\frac{\pi}{2}}$$
$$=\lim_{x\to 0}\frac{\sqrt{1-\cos x}}{-x}=\lim_{x\to 0}-\sqrt{\frac{1-\cos x}{x^2}}\cdot\operatorname{sgn}(x).$$

This limit does not exist, since the limit from the right equals $-\frac{1}{\sqrt{2}}$ and the limit from the left equals $\frac{1}{\sqrt{2}}$.
Similarly for the other points.
Approximate evaluation:
Determining the domain: 2 points.
Picture of the domain: 1 point.
Formulas for the partial derivatives: 2 points.
Domain of the validity of these formulas: 1 point.
Determining in which of the remaining points the partial derivatives can be defined: 2 points.
Computation of the respective limits: 2 points (1 point each).

Problem 3: Let us consider the equation

$$\cos(x+y^2)+\sin(x^2+y)=1$$

and the point $[-1,-1]$. Show that this equation defines a C^∞ function $y=f(x)$ defined on a neighborhood of -1, which satisfies $f(-1)=-1$. Compute $f'(-1)$, $f''(-1)$ and determine the equation of the tangent line to the graph of f at the point $[-1,f(-1)]$.

Sketch of solution. Denote $F(x,y)=\cos(x+y^2)+\sin(x^2+y)-1$. To show the existence of f with the given properties, it is enough to check the assumptions of the implicit function theorem:
(1) F is C^∞ on an open set containing $[-1,-1]$. This is true, because $F\in C^\infty(\mathbb{R}^2)$. This follows from the properties of the elementary functions.
(2) $F(-1,-1)=0$. This can be checked by computation.
(3) $\frac{\partial F}{\partial y}(-1,-1)\neq 0$. Indeed,

$$\frac{\partial F}{\partial y}(-1,-1)=-sin(x+y^2)\cdot 2y+\cos(x^2+y)\big|_{x=-1,y=-1}=1.$$

The implicit function theorem then yields the existence of f with the required properties. In particular, f satisfies

$$\cos(x+f(x)^2)+\sin(x^2+f(x))-1=0 \qquad \text{on a neighborhood of } -1.$$

To compute the first derivative, we differentiate this equation:

$$-\sin(x+f(x)^2)\cdot(1+2f(x)f'(x))+\cos(x^2+f(x))\cdot(2x+f'(x))=0 \qquad \text{on a neighborhood of } -1.$$

If we substitute there $x=-1$ and use that $f(-1)=-1$, we get

$$-\sin(0)\cdot(1-2f'(-1))+\cos(0)\cdot(-2+f'(-1))=0,$$

hence $f'(-1)=2$.
To compute the second derivative, we differentiate once more. We get

$$\begin{aligned}&-\cos(x+f(x)^2)\cdot(1+2f(x)f'(x))^2-\sin(x+f(x)^2)\cdot(2(f'(x))^2+2f(x)f''(x))\\ &-\sin(x^2+f(x))\cdot(2x+f'(x))^2+\cos(x^2+f(x))\cdot(2+f''(x))=0 \qquad \text{on a neighborhood of } -1.\end{aligned}$$

Further, we substitute $x=-1$ and use the fact $f(-1)=-1$ and the already computed value $f'(-1)=2$. We get

$$-\cos(0)\cdot(1+2\cdot(-1)\cdot 2)^2-\sin(0)\cdot(2\cdot 2^2+4f''(-1))-\sin(0)\cdot(-2+2))^2+\cos(0)\cdot(2+f''(-1))=0,$$

hence $f''(-1)=7$.
Equation of the tangent plane is then $y=-1+2(x+1)$.

Approximate evaluation:
Checking the assumptions of IFT: 3 points
The equation satisfied by f: 1 point
Differentiating the equation for the first time: 1 point
Computing $f'(-1)$: 1 point
Differentiating the equation for the second time: 2 points
Computing $f''(-1)$: 1 point
Tangent line: 1 point

Problem 4: Determine sup and inf of the function f on the set M and decide whether these values are attained, if

$$f(x,y,z) = x^2y \text{ and } M = \{[x,y,z] \in \mathbb{R}^3 : x^2+y^2+2z^2 = 4, xz \geq 1\}$$

Sketch of solution:
Existence of extrema: f is continuous on $\mathbb{R}^3$. The set M is closed (it is determined by equality and non-strict inequality of continuous functions) and bounded (it is contained in the closed ball centered at origin with radius 2. Therefore it is compact. Thus the extrema (minimum and maximum) do exist provided M is nonempty. This assumption will be checked later by finding some points in M.
To find extrema, we decompose M to two sets, $M = M_1 \cup M_2$, where

$$M_1 = \{[x,y,z] \in \mathbb{R}^3 : x^2+y^2+2z^2 = 4, xz > 1\},$$
$$M_2 = \{[x,y,z] \in \mathbb{R}^3 : x^2+y^2+2z^2 = 4, xz = 1\}.$$

Further, denote $g_1(x) = x^2+y^2+2z^2-4$ and $g_2(x,y,z) = xz-1$. For future use let us compute gradients:

$$\nabla f(x,y,z) = [2xy, x^2, 0],$$
$$\nabla g_1(x,y,z) = [2x, 2y, 4z],$$
$$\nabla g_2(x,y,z) = [z, 0, x].$$

Let us now investigate the set M_1. By the Lagrange multiplier theorem, if at some point $[x,y,z] \in M_1$ there is a local extremum of f with respect to M_1, then at this point either $\nabla g_1 = \mathbf{o}$ or there is $\lambda \in \mathbb{R}$ such that $\nabla f + \lambda \cdot \nabla g_1 = 0$.
Let us first consider the first possibility. $\nabla g_1 = \mathbf{o}$ only at the point $[0,0,0]$, but this point does not belong to M_1 (it does not satisfy the respective equation). So, this possibility is excluded.
So, in points of extrema there is $\lambda \in \mathbb{R}$ such that

$$2xy + \lambda \cdot 2x = 0,$$
$$x^2 + \lambda \cdot 2y = 0,$$
$$0 + \lambda \cdot 4z = 0$$

The third equation implies that either $z = 0$ or $\lambda = 0$. If $z = 0$, then also $xz = 0$ and such points cannot be in M_1 since all the points of M_1 satisfy $xz > 1$. If $\lambda = 0$, then the second equation yields $x = 0$, which is again in contradiction with $xz > 1$. Therefore there are no extrema in M_1.
So, let us continue with M_2. By the Lagrange multiplier theorem, if at some point $[x,y,z] \in M_2$ there is a local extremum of f with respect to M_2, then at this point either ∇g_1 and ∇g_2 are linearly dependent or there are $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\nabla f + \lambda \cdot \nabla g_1 + \lambda_2 \nabla g_2 = \mathbf{o}$.
Let us first consider the first possibility. If ∇g_1 and ∇g_2 are linearly dependent, then either there is $\alpha \in \mathbb{R}$ such that $\nabla g_1 = \alpha \nabla g_2$ or there is $\alpha \in \mathbb{R}$ such that $\nabla g_2 = \alpha \nabla g_1$.
If $\nabla g_1 = \alpha \nabla g_2$, i.e. $[2x, 2y, 4z] = [\alpha z, 0, \alpha x]$, necessarily $y = 0$. Further, we get $2x = \alpha z = \alpha \cdot \frac{1}{4}\alpha x = \frac{1}{4}\alpha^2 x$. It follows that either $x = 0$ or $\alpha^2 = 8$. If $x = 0$, then necessarily $z = 0$, which is not possible as $[0,0,0] \notin M_2$. Hence $\alpha^2 = 8$, so $x^2 = 2z^2$. Hence, from the first equation defining M_2 we get $2z^2 + 2z^2 = 4$, thus $z^2 = 1$ and $x^2 = 2$. But it is in contradiction with the second equation $xz = 1$. So, the possibility $\nabla g_1 = \alpha \nabla g_2$ is excluded.
Suppose now $\nabla g_2 = \alpha \nabla g_1$. If $\alpha \neq 0$, we get $\nabla g_1 = \frac{1}{\alpha} \nabla g_2$, which is impossible by the previous paragraph. If $\alpha = 0$, we get $x = z = 0$. But it contradicts the equation $xz = 1$, so no such point belongs to M_2.
Finally, suppose that there are $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\nabla f + \lambda \cdot \nabla g_1 + \lambda_2 \nabla g_2 = \mathbf{o}$. I.e., we have the following system of equations:

$$2xy + \lambda_1 \cdot 2x + \lambda_2 z = 0,$$
$$x^2 + \lambda_1 \cdot 2y + \lambda_2 \cdot 0 = 0,$$
$$0 + \lambda_1 \cdot 4z + \lambda_2 x = 0$$

If we multiply the first equation by x and the third one by z and subtract the resulting two equations, we obtain

$$2x^2y + \lambda_1(2x^2 - 4z^2) = 0.$$

If we now multiply this equation by y, the second equation by $x^2 - 2z^2$ and subtract the resulting equations, we get

$$2x^2y^2 - x^2(x^2 - 2z^2) = 0.$$

Since $x \neq 0$ due the the equation $xz = 1$, the above equation can be divided by x^2, so we get

$$2y^2 - x^2 + 2z^2 = 0, \qquad hencey^2 = \frac{1}{2}x^2 - z^2.$$

If we substitute this into the first equation defining M_2, we obtain

$$\frac{3}{2}x^2 + z^2 = 4.$$

If we substitute there moreover $z = \frac{1}{x}$, we get

$$\frac{3}{2}x^2 + \frac{1}{x^2} = 4, \text{ hence } \frac{3}{2}x^4 - 4x^2 + 1.$$

This is a biquadratic equation, so

$$x^2 = \frac{4 \pm \sqrt{10}}{3}, \text{ hence } x = \pm\sqrt{\frac{4 \pm \sqrt{10}}{3}},$$

where all the four combinations of signs are allowed.
Now we compute $z = \frac{1}{x}$ and $y^2 = \frac{1}{2}x^2 - z^2$. In case $x = \pm\sqrt{\frac{4+\sqrt{10}}{3}}$, we obtain in this way four points, in case $x = \pm\sqrt{\frac{4-\sqrt{10}}{3}}$ no solutions are obtained since one gets $y^2 < 0$ which is impossible.
If we compute the values of f at the four obtained points and compare them, we get the results: Maximum $\frac{2}{9}\sqrt{3}\sqrt{14 + 5\sqrt{10}}$ at points $\left[\pm\sqrt{\frac{4+\sqrt{10}}{3}}, \sqrt{\frac{2\sqrt{10}-3}{3}}, \pm\sqrt{\frac{3}{4+\sqrt{10}}}\right]$ (the two signs are the same), minimum $-\frac{2}{9}\sqrt{3}\sqrt{14 + 5\sqrt{10}}$ at points $\left[\pm\sqrt{\frac{4+\sqrt{10}}{3}}, -\sqrt{\frac{2\sqrt{10}-3}{3}}, \pm\sqrt{\frac{3}{4+\sqrt{10}}}\right]$ (the two signs are the same).

Approximate evaluation:
Existence of extrema: 2 points
Decomposition to $M_1 \cup M_2$: 1 point
On M_1:
- the case of the zero gradient 1 point
- the stating of the equations with mutlipliers 1 point
- solving the equations 2 points

On M_2:
- the case of linearly dependent gradients 2 points
- the stating of the equations with mutlipliers 1 points
- solving the equations 3 points

Final discussion: 2 points

Problem 5: Compute the following antiderivative on maximal possible intervals:

$$\int \frac{x^4 + 4x^3 + x^2 + x + 1}{(x^2 + x + 1)(x^2 + 5x + 4)}\,\mathrm{d}x$$

Sketch of solution:
The integrand is a rational function. The degrees of nominator and denominator are equal to 4, so we should start by division of the two polynomials.
The denominator can be expanded to $x^4 + 6x^3 + 10x^2 + 9x + 4$, hence we have

$$\frac{x^4 + 4x^3 + x^2 + x + 1}{(x^2 + x + 1)(x^2 + 5x + 4)} = \frac{x^4 + 4x^3 + x^2 + x + 1}{x^4 + 6x^3 + 10x^2 + 9x + 4} = 1 + \frac{-2x^3 - 9x^2 - 8x - 3}{x^4 + 6x^3 + 10x^2 + 9x + 4}.$$

The second fraction is a rational function where the degree of the nominator is strictly smaller than the degree of the denominator, so the next step is the decomposition to the partial fractions. First we need to decompose the denominator to a suitable product. We have

$$x^4 + 6x^3 + 10x^2 + 9x + 4 = (x^2 + x + 1)(x^2 + 5x + 4) = (x^2 + x + 1)(x + 4)(x + 1),$$

where the first step follows from the fact that left-hand side has been computed by expanding the respective product. Further, the quadratic polynomial $x^2 + x + 1$ has no real roots, so the above decomposition is the required one. Now, to decompose the fraction to partial fractions we need to find $A, B, C, D \in \mathbb{R}$ such that

$$\frac{-2x^3 - 9x^2 - 8x - 3}{x^4 + 6x^3 + 10x^2 + 9x + 4} = \frac{Ax + B}{x^2 + x + 1} + \frac{C}{x + 4} + \frac{D}{x + 1}.$$

If we multiply this equation by the denominator of the left-hand side, we get

$$-2x^3 - 9x^2 - 8x - 3 = (Ax + B)(x + 4)(x + 1) + C(x^2 + x + 1)(x + 1) + D(x^2 + x + 1)(x + 4).$$

If we substitute there $x = -1$, we get

$$-2 = 3D, \text{ hence } D = -\frac{2}{3}.$$

If we substitute $x = -4$, we get

$$13 = -39C, \text{ hence } C = -\frac{1}{3}.$$

To find the values of A and B, we will compare the coefficient of the polynomials on the right-hand side and on the left-hand side. The coefficient at x^3 of the right-hand side is $A + C + D = A - 1$. We thus get

$$-2 = A - 1, \text{ hence } A = -1.$$

Further, the coefficient at x^0 on the right-hand side is $4B + C + 4D = 4B - 3$. We thus get

$$-3 = 4B - 3, \text{ hence } B = 0.$$

We therefore have

$$\frac{-2x^3 - 9x^2 - 8x - 3}{x^4 + 6x^3 + 10x^2 + 9x + 4} = \frac{-x}{x^2 + x + 1} + \frac{-\frac{1}{3}}{x + 4} + \frac{-\frac{2}{3}}{x + 1}.$$

The first fraction should be furhter decomposed:

$$\frac{-x}{x^2 + x + 1} = -\frac{1}{2}\frac{2x + 1}{x^2 + x + 1} + \frac{\frac{1}{2}}{x^2 + x + 1}.$$

The antiderivative to the first fraction is $-\frac{1}{2}\log(x^2+x+1)$. The antiderivative to the second one is computed in the standard way:

$$\int \frac{\frac{1}{2}}{x^2+x+1}\,\mathrm{d}x = \int \frac{1}{2}\cdot\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,\mathrm{d}x = \int \frac{1}{2}\cdot\frac{4}{3}\frac{1}{\frac{4}{3}(x+\frac{1}{2})^2+1}\,\mathrm{d}x$$

$$= \int \frac{2}{3}\cdot\frac{\frac{\sqrt{3}}{2}\cdot\frac{2}{\sqrt{3}}}{(\frac{2}{\sqrt{3}}(x+\frac{1}{2}))^2+1}\,\mathrm{d}x \stackrel{c}{=} \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} \quad \text{on } \mathbb{R}.$$

By putting together the partial results we get that

$$\int \frac{x^4+4x^3+x^2+x+1}{(x^2+x+1)(x^2+5x+4)}\,\mathrm{d}x \stackrel{c}{=} x-\frac{1}{2}\log(x^2+x+1)+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}-\frac{1}{3}\log|x+4|-\frac{2}{3}\log|x+1|$$

on each of the three intervals $(-\infty,-4)$, $(-4,-1)$ and $(-1,+\infty)$.

Approximate evaluation:
Division of polynomials: 1 point
Decomposing to the partial fractions:
- correct statement of the general equation 1 point
- solution 3 points
Integration of the first fraction:
- further decomposition 1 point
- integration of the first part 1 point
- integration of the second part 3 points
Integration of the second and third fractions: 2 points
Putting the results together: 1 point
Determining of the intervals: 2 points