Věta 24 (a) $f \in L^1_{loc}(\mathbb{R}^d) \Rightarrow \Lambda_f * \varphi = f * \varphi$

$$\ulcorner\ \Lambda_f * \varphi\,(x) = \Lambda_f(\tau_x \check{\varphi}) = \int_{\mathbb{R}^d} f(y) \cdot \tau_x \check{\varphi}(y)\,dy =$$

$$= \int_{\mathbb{R}^d} f(y) \cdot \varphi(x-y)\,dy = f * \varphi(x)\ \lrcorner$$

(b) $U * \varphi \in C^\infty(\mathbb{R}^d)$ a $\forall \alpha:\ D^\alpha(U * \varphi) = (D^\alpha U) * \varphi = U * D^\alpha \varphi$

$$\ulcorner\ U * \varphi\,(x) = U(y \mapsto \varphi(x-y))$$

$$U * D^\alpha \varphi\,(x) = U(y \mapsto D^\alpha \varphi(x-y))$$

$$D^\alpha U * \varphi\,(x) = D^\alpha U(y \mapsto \varphi(x-y)) = (-1)^{|\alpha|} U(D^\alpha(y \mapsto \varphi(x-y))) =$$
$$= (-1)^{|\alpha|} U(y \mapsto (-1)^{|\alpha|} D^\alpha \varphi(x-y)) =$$
$$= U(y \mapsto D^\alpha \varphi(x-y)) = U * D^\alpha \varphi\,(x)$$

Spočteme $\frac{\partial}{\partial x_1} U * \varphi(x) = \lim_{r \to 0} \frac{U * \varphi(x + re^1) - U * \varphi(x)}{r} =$

$$= \lim_{r \to 0} \frac{1}{r} U\left(y \mapsto (\varphi(x + re^1 - y) - \varphi(x-y))\right) =$$

$$= \lim_{r \to 0} U\Big(\underbrace{y \mapsto \frac{\varphi(x + re^1 - y) - \varphi(x-y)}{r}}_{\text{dle L22(b)}\ \xrightarrow{\ v\ \mathcal{D}(\mathbb{R}^d)\ }\ y \mapsto \frac{\partial}{\partial x_1}\varphi(x-y)}\Big) = U\left(y \mapsto \frac{\partial}{\partial x_1}\varphi(x-y)\right) = \left(U * \frac{\partial \varphi}{\partial x_1}\right)(x)$$

Nyní stačí ukázat, že $U * \varphi$ je spojitá a použít indukci podle α

K tomu stačí: $x_n \to x$ v $\mathbb{R}^d \Rightarrow$
„$(y \mapsto \varphi(x_n - y))$" $\to$ „$(y \mapsto \varphi(x-y))$" v $\mathcal{D}(\mathbb{R}^d)$
což už víme z Lemmatu 22(a)

(c) $\operatorname{supp}(U*\varphi) \subset \operatorname{supp} U + \operatorname{supp}\varphi$

⌈ $\operatorname{supp} U$ uzavřený, $\operatorname{supp}\varphi$ kompaktní $\Rightarrow$ $\operatorname{supp} U + \operatorname{supp}\varphi$ uzavřená

$x \notin \operatorname{supp} U + \operatorname{supp}\varphi \Rightarrow \exists\, R>0 : B(x,r) \cap (\operatorname{supp} U + \operatorname{supp}\varphi) = \emptyset$

$U*\varphi(x) = U(y \mapsto \varphi(x-y))$

$\varphi(x-y) \neq 0 \Rightarrow x-y \in \operatorname{supp}\varphi \Rightarrow y \in \cancel{\operatorname{supp}} \; x - \operatorname{supp}\varphi$

tj $\operatorname{supp}(y \mapsto \varphi(x-y)) \subset x - \operatorname{supp}\varphi$

aby $U*\varphi(x) \neq 0$, musí ~~...~~ mezi protínat $\operatorname{supp} U$,

tzn $\operatorname{supp} U \cap (x - \operatorname{supp}\varphi) \neq \emptyset \Rightarrow x \in \operatorname{supp} U + \operatorname{supp}\varphi$

Tedy ~~...~~ opravdu $\operatorname{supp} U*\varphi \subset \operatorname{supp} U + \operatorname{supp}\varphi$ ⌋

(d) (h_j) aprox. jednotky $\Rightarrow$ $\Lambda_{U*h_j} \longrightarrow U$ v $\mathcal{D}'(\mathbb{R}^d)$

⌈ $\varphi \in \mathcal{D}(\mathbb{R}^d) \Rightarrow$

$$\Lambda_{U*h_j}(\varphi) = \int \varphi(x)\, U*h_j(x)\, dx =$$

$$= \int \varphi(x) \cdot U(y \mapsto h_j(y-x))\, dx =$$

$$= \int U(y \mapsto \varphi(x) h_j(y-x))\, dx =$$

[$(x,y) \mapsto \varphi(x) h_j(y-x)$ je třídy C^∞ na $\mathbb{R}^d \times \mathbb{R}^d$

$\varphi(x) h_j(y-x) \neq 0 \Rightarrow x \in \operatorname{supp}\varphi$

$y - x \in \operatorname{supp} h_j \Rightarrow y \in \operatorname{supp} h_j + x \subset \operatorname{supp}\varphi + \operatorname{supp} h_j$

$\Rightarrow (x,y) \in \operatorname{supp}\varphi \times (\operatorname{supp}\varphi + \operatorname{supp} h_j)$]

Tvrz 23

$$= \Lambda_1(x \mapsto U(y \mapsto \varphi(x) h_j(y-x))) \overset{\text{Tvrz 23}}{=} U(y \mapsto \Lambda_1(x \mapsto \varphi(x) h_j(y-x)))$$

$$= U(y \mapsto \varphi * h_j(y)) = U(\varphi * h_j) \longrightarrow U(\varphi) \text{ protože } \varphi * h_j \to \varphi \text{ v } \mathcal{D}(\mathbb{R}^d)$$

e) $\tau_x(U * \varphi) = (\tau_x U) * \varphi = U * \tau_x \varphi$

$\ulcorner$ $\tau_x(U*\varphi)(y) = U*\varphi(y-x) = U(z \mapsto \varphi(y-x-z))$

$((\tau_x U) * \varphi)(y) = \tau_x U(z \mapsto \varphi(y-z)) = U(\tau_{-x}(z \mapsto \varphi(y-z)))$
$= U(z \mapsto \varphi(y-(z+x)))$

$U * \tau_x \varphi(y) = U(z \mapsto \tau_x\varphi(y-z)) = U(z \mapsto \varphi(y-z-x))$ $\lrcorner$

f) $((U*\varphi)*\psi = U*(\varphi*\psi)$

$(U*\varphi)*\psi(x) = \int_{\mathbb{R}^d} U*\varphi(x-y)\cdot\psi(y)\,dy = \int_{\mathbb{R}^d} U(z \mapsto \varphi(x-y-z))\psi(y)\,dy$

$= \int_{\mathbb{R}^d} U(z \mapsto \varphi(x-y-z)\psi(y))\,dy =$

$\ulcorner$ $(y,z) \mapsto \varphi(x-y-z)\psi(y)$ je C^∞ na $\mathbb{R}^d \times \mathbb{R}^d$

pisać $\neq 0 \Rightarrow$ • $y \in \operatorname{supp}\psi$

• $x-y-z \in \operatorname{supp}\varphi \Rightarrow z \in x-y-\operatorname{supp}\varphi \subset x-\operatorname{supp}\psi-\operatorname{supp}\varphi$

Twier '23 — Tej je to z $\mathcal{D}(\mathbb{R}^d \times \mathbb{R}^d)$ $\lrcorner$

$= \Lambda_1(y \mapsto U(z \mapsto \varphi(x-y-z)\psi(y))) = U(z \mapsto \Lambda_1(y \mapsto \varphi(x-y-z)\psi(y)))$

$= U(z \mapsto \varphi*\psi(x-z)) = U*(\varphi*\psi)(x)$