Veta IV.25

(a) $U, V \in \mathcal{D}'(\mathbb{R}^d)$, V má kompaktní nosič

$$U * V(\varphi) = (U * (V * \check{\varphi}))(0) = U(\check{V} * \varphi)$$

• dále definovaná: V má komp. nosič $\Rightarrow$ $V * \check{\varphi} \in \mathcal{D}(\mathbb{R}^d)$
$\check{V} * \varphi \in \mathcal{D}(\mathbb{R}^d)$

$$U * (V * \check{\varphi})(0) = U(y \mapsto V * \check{\varphi}(0-y)) = U(\check{(V * \check{\varphi})})$$

Přitom $\check{(V * \varphi)} = \check{V} * \check{\varphi}$

$$(\check{(V * \varphi)}(x) = V * \varphi(-x) = V(y \mapsto \varphi(-x-y)) = \check{V}(y \mapsto \varphi(-x+y))$$

$$= \check{V}(y \mapsto \check{\varphi}(x-y)) = \check{V} * \check{\varphi}(x) \quad)$$

Tedy opravdu $(U * (V * \check{\varphi}))(0) = U(\check{V} * \varphi)$

Je to distribuce:

- linearita zřejmá
- spojitost:
  - V má kompaktní nosič $\Rightarrow \exists N_1, C_1$, že $|V(\varphi)| \leq C_1 \|\varphi\|_{N_1}$ pro každé $\varphi \in \mathcal{D}(\mathbb{R}^d)$
  - $K \subset \mathbb{R}^d$ kompaktní $\Rightarrow L := \operatorname{spt} \check{V} + K$ též kompaktní a $\operatorname{spt} \varphi \subset K \Rightarrow \operatorname{spt}(\check{V} * \varphi) \subset L$
    ex. N_2, C_2, že $|U(\varphi)| \leq C_2 \|\varphi\|_{N_2}$ pro $\varphi \in \mathcal{D}_L(\mathbb{R}^d)$

$$\varphi \in \mathcal{D}_K(\mathbb{R}^d) \Rightarrow \check{V} * \varphi \in \mathcal{D}_L(\mathbb{R}^d) \Rightarrow |(U * V)(\varphi)| = |U(\check{V} * \varphi)| \leq C_2 \|\check{V} * \varphi\|_{N_2}$$

$$|\alpha| \leq N_2 \Rightarrow |D^\alpha(\check{V} * \varphi)(x)| = |(\check{V} * D^\alpha \varphi)(x)| = |\check{V}(y \mapsto D^\alpha \varphi(x-y))| =$$

$$= |V(y \mapsto D^\alpha \varphi(x+y))| \leq C_1 \|y \mapsto D^\alpha \varphi(x+y)\|_{N_1} \leq C_1 \|\varphi\|_{N_1+N_2}$$

$$\Rightarrow |U * V(\varphi)| \leq C_2 \cdot C_1 \cdot \|\varphi\|_{N_1+N_2}$$

$$V * U(\varphi) = V(\psi \cdot (\check{U} * \varphi))$$, kde $\psi \in \mathcal{D}(\mathbb{R}^d)$, $\psi = 1$ na okolí množiny obsahující spt V

Pak $V * U = U * V$

$$V * U(\varphi) = V(x \mapsto \psi(x) \cdot (\check{U} * \varphi)(x)) = V(x \mapsto \psi(x)\, \check{U}(y \mapsto \varphi(x-y)))$$
$$= V(x \mapsto U(y \mapsto \psi(x)\varphi(x+y))) =$$

$\ulcorner$ $(x, y) \mapsto \psi(x)\varphi(x+y)$ je třídy C^∞

a posledně $\mathcal{O}$, pro $x \in \operatorname{spt} \psi$

$x+y \in \operatorname{spt}\varphi \Rightarrow y \in \operatorname{spt}\varphi - x \subset \operatorname{spt}\varphi - \operatorname{spt}\psi$

T 23 $\Rightarrow$ je to v $\mathcal{D}(\mathbb{R}^d \times \mathbb{R}^d)$ $\lrcorner$

$\Downarrow$

$$= U(y \mapsto V(x \mapsto \psi(x)\varphi(x+y))) = U(y \mapsto V(x \mapsto \varphi(x+y)))$$

$$= U(y \mapsto \check{V}(x \mapsto \varphi(y-x))) = U(\check{V} * \varphi) = U * V(\varphi) \lrcorner$$

(b)

$$U * \Lambda_\varphi(\psi) = U(\check{\Lambda}_\varphi * \psi) = U(\Lambda_{\check\varphi} * \psi) = U(\check\varphi * \psi) =$$

$$= U\Big(x \mapsto \underbrace{\int_{\mathbb{R}^d} \check\varphi(x-y)\,\psi(y)\,dy}_{\in \mathcal{D}(\mathbb{R}^d \times \mathbb{R}^d)}\Big) = U\big(x \mapsto \Lambda_1(y \mapsto \check\varphi(x-y)\,\psi(y))\big) =$$

[je tudi C^∞
podpore to $\neq 0$, ga
$y \in \operatorname{sp}\psi$,
$x-y \in \operatorname{sp}\check\varphi = -\operatorname{sp}\varphi$
$x \in y - \operatorname{sp}\varphi \subset \operatorname{sp}\psi - \operatorname{sp}\varphi$]

$$\overset{T23}{=} \Lambda_1\big(y \mapsto U(x \mapsto \check\varphi(x-y)\,\psi(y))\big) =$$

$$= \int_{\mathbb{R}^d} \psi(y) \cdot U(x \mapsto \varphi(y-x))\,dy =$$

$$= \int_{\mathbb{R}^d} \psi(y) \cdot U*\varphi(y)\,dy = \Lambda_{U*\varphi}(\psi)$$

(c) $f \in L^1_{loc}(\mathbb{R}^d),\ \varphi \in \mathcal{D}(\mathbb{R}^d) \Rightarrow \Lambda_f * \Lambda_\varphi = \Lambda_{f*\varphi}$

[$\Lambda_f * \Lambda_\varphi \overset{(b)}{=} \Lambda_{\Lambda_f * \varphi} = \Lambda_{f*\varphi}$]

(d) $\operatorname{sp}(U*V) \subset \operatorname{sp}U + \operatorname{sp}V$

[BŠNO V ima kompakten nosilec

$U*V(\varphi) \neq 0 \Rightarrow U(\check V * \varphi) \neq \emptyset \Rightarrow$

$\operatorname{sp}(\check V * \varphi) \cap \operatorname{sp}U \neq \emptyset \Rightarrow (\operatorname{sp}\check V + \operatorname{sp}\varphi) \cap \operatorname{sp}U \neq \emptyset$

$\qquad\qquad\qquad\qquad\qquad \| \ (-\operatorname{sp}V + \operatorname{sp}\varphi)$

Tog $\operatorname{sp}\varphi \cap (\operatorname{sp}U + \operatorname{sp}V) \neq \emptyset$. A to rabmo.]

(e) $D^\alpha(U*V) = D^\alpha U * V = U * D^\alpha V$

[BŠNO V ima komp. nosilec

$D^\alpha(U*V)(\varphi) = (-1)^{|\alpha|}\,U*V(D^\alpha\varphi) = (-1)^{|\alpha|}\,U(\check V * D^\alpha\varphi) \overset{(*)}{=}$

$= (-1)^{|\alpha|}\,U(D^\alpha(\check V * \varphi)) = D^\alpha U(\check V * \varphi) = (D^\alpha U * V)(\varphi)$

$(*) = (-1)^{|\alpha|}\,U(D^\alpha\check V * \varphi) = U((-1)^{|\alpha|}D^\alpha\check V * \varphi) = U(\widecheck{D^\alpha V} * \varphi) =$

$= U * D^\alpha V(\varphi)$]

(f) $U = U * 1_{\delta_0}$, $D^\alpha U = U * D^\alpha 1_{\delta_0}$

• $(U * 1_{\delta_0})(\varphi) = U(\check{1}_{\delta_0} * \varphi) = U(1_{\delta_0} * \varphi) = U(\varphi)$

$\check{1}_{\delta_0}(\varphi) = 1_{\delta_0}(\check{\varphi}) = \check{\varphi}(0) = \varphi(0)$

$1_{\delta_0} * \varphi(t) = 1_{\delta_0}(y \mapsto \varphi(t-y)) = \varphi(t-0) = \varphi(t)$

• $U * D^\alpha 1_{\delta_0} = D^\alpha U * 1_{\delta_0} = D^\alpha U$

(g) W má komp. nosič $\Rightarrow$ $U * (V * W) = (U * V) * W$

Případ 1: V má komp. nosič. Pak

$U * (V * W)(\varphi) = U(\widecheck{V * W} * \varphi) = U((\check{V} * \check{W}) * \varphi)$

$(U * V) * W(\varphi) = (U * V)(\check{W} * \varphi) = U(\check{V} * (\check{W} * \varphi))$

Stačí tedy: V, W mají komp. nosič, $\varphi \in \mathcal{D}(\mathbb{R}^d) \Rightarrow (V * W) * \varphi = V * (W * \varphi)$

$(V * W) * \varphi(t) = V * W(y \mapsto \varphi(x-y)) = V(\check{W} * (y \mapsto \varphi(t-y))) =$

$= V(y \mapsto \check{W}(z \mapsto \varphi(t-(y-z)))) = V(y \mapsto W(z \mapsto \varphi(t-y-z)))$

$V * (W * \varphi)(t) = V(y \mapsto W * \varphi(t-y)) = V(y \mapsto W(z \mapsto \varphi(t-y-z)))$

Vyšlo to stejně.

Případ 2: U má komp. nosič. S využitím komutativity:

$U * (V * W)(\varphi) = (V * W) * U(\varphi) = V * W(\check{U} * \varphi) = V(\check{W} * (\check{U} * \varphi))$

$(U * V) * W(\varphi) = U * V(\check{W} * \varphi) = V * U(\check{W} * \varphi) = V(\check{U} * (\check{W} * \varphi))$

Dle případu 1 je $\check{W} * (\check{U} * \varphi) = (\check{W} * \check{U}) * \varphi = (\check{U} * \check{W}) * \varphi = \check{U} * (\check{W} * \varphi)$.