Lemma 30 $\phi_d(t) = e^{-\frac{1}{2}\|t\|^2}$, $t \in \mathbb{R}^d$. Pak $\phi_d \in \mathcal{S}(\mathbb{R}^d)$

a $\widehat{\phi_d} = \phi_d$

Dk: ① $\phi_d \in \mathcal{S}$, protože $\forall \alpha : D^\alpha \phi_d = P \cdot \phi_d$, kde P je polynom

a $\forall P$ polynom: $\lim_{\|t\| \to \infty} P(x) e^{-\frac{1}{2}\|t\|^2} = 0$

② $d = 1$ $\Rightarrow$ $\phi_1' = -t \cdot \phi_1$

proto $(\widehat{\phi_1})'(t) = \widehat{-ix\phi_1(x)}(t) = i \cdot \widehat{\phi_1'}(t) = i \cdot i t \widehat{\phi_1}(t)$

$= -t\widehat{\phi_1}(t)$

$\Rightarrow \phi_1$ i $\widehat{\phi_1}$ řeší diferenciální rovnici $y' + ty = 0$

navíc $\phi_1(0) = 1$

$\widehat{\phi_1}(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{t^2}{2}} dx = 1$

$\Rightarrow \widehat{\phi_1} = \phi_1$

③ $\widehat{\phi_d}(t) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-\|t\|^2/2} \cdot e^{-i\langle t, x\rangle} dx =$

$= \prod_{j=1}^{d} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} e^{-x_j^2/2} \cdot e^{-i t_j x_j} dx_j \overset{②}{=} \prod_{j=1}^{d} e^{-t_j^2/2} = e^{-\|t\|^2/2}$

$\phi_d(t)$

V31 (a) $f \in \mathcal{S} \Rightarrow$ za $\lambda > 0$

$$\int_{\mathbb{R}^d} f\left(\frac{t}{\lambda}\right) \phi_d(t)\, dm_d(t) = \int_{\mathbb{R}^d} f(y)\, \phi_d(\lambda y) \cdot \lambda^d \, dm_d(t) =$$

$$= \int_{\mathbb{R}^d} f(y) \cdot \lambda^d \hat{\phi}_d(\lambda y)\, dm_d(t) = \int_{\mathbb{R}^d} f(y)\, \widehat{\phi_d\left(\frac{y}{\lambda}\right)}\, dm_d(y) =$$

$$= \int_{\mathbb{R}^d} \hat{f}(y) \cdot \phi_d\left(\frac{y}{\lambda}\right) dm_d(y)$$

$$\lambda \to \infty \Rightarrow \int_{\mathbb{R}^d} f(0)\, \phi_d(t)\, dm_d(t) = \int_{\mathbb{R}^d} \hat{f}(y)\, \phi_d(0)\, dm_d(y)$$

Leb. izrek. (integr. majoranta $C \phi_d$ oz. $\hat{f}$, konst.)

$$\Rightarrow f(0) = \int_{\mathbb{R}^d} \hat{f}\, dm_d$$

$$f(t) = (\tau_{-t} f)(0) = \int_{\mathbb{R}^d} \widehat{(\tau_{-t} f)}\, dm_d = \int_{\mathbb{R}^d} \hat{f} \cdot e_t \, dm_d,$$

kar je bilo treba.

(b) $$\hat{\hat{f}}(t) = \int_{\mathbb{R}^d} \hat{f}(t) e^{-i\langle \xi, t \rangle} = \int_{\mathbb{R}^d} \hat{f}(t) e^{i\langle t, -t \rangle}\, dm_d(t) \overset{(a)}{=} f(-t) = \check{f}(t)$$

Torej: $\hat{\hat{\hat{f}}} = \check{\hat{f}} = f$

$\mathcal{F}: f \mapsto \hat{f}$ je surjekcija, inverz $\mathcal{F}^{-1}(f) = \check{\hat{f}}$, odtod izpeljan $\mathcal{F}^{-1}$

Dnoslcdek 32: Naj bo $f \in C^1(\mathbb{R}^d)$ in $\hat{f} \in C^1(\mathbb{R}^d)$

$$\Rightarrow f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{i\langle \xi, x\rangle} \, dm_d(\xi) \quad \text{s.v.}$$

D4: Ta računamo sedaj vyrazit pak:

$$\int_{\mathbb{R}^d} \hat{f}(\xi) e^{i\langle \xi, x\rangle} \, dm_d(\xi) = \int_{\mathbb{R}^d} \hat{f}(-\xi) e^{-i\langle -\xi, x\rangle} \, dm_d(\xi) =$$

$$= \check{\hat{f}}(x)$$

Pro každé $g \in \mathcal{S}(\mathbb{R}^d)$ platí:

$$\int f \cdot \hat{g} \, dm_d \underset{T26(g)}{=} \int \hat{f} \cdot g \underset{V31(3)}{=} \int \hat{f} \cdot \hat{\check{g}} \underset{\substack{\text{substituce} \\ x \mapsto -x}}{=} \int \check{\hat{f}} \, \hat{g} \underset{T26(g)}{=} \int \hat{\check{\hat{f}}} \cdot \hat{g}$$

$$\Rightarrow \int (f - \check{\hat{f}}) \hat{g} = 0 \quad \text{pro vš. } g \in \mathcal{S}$$

$f \in C^1$, $\check{\hat{f}} \in C_0 \Rightarrow f - \check{\hat{f}} \in L^1_{loc}$. Protože $\mathcal{S} \supset \mathcal{D}(\mathbb{R}^d)$,

je $\Lambda_{f - \check{\hat{f}}} = 0 \underset{L6}{\Rightarrow} f - \check{\hat{f}} = 0$ s.v. Atd. poměno