Tvrzení 33 $\quad f, g \in \mathcal{S} \Rightarrow \widehat{f \cdot g} = \hat{f} * \hat{g}$

Γ $\hat{f}, \hat{g} \in \mathcal{S} \subset L^1(\mathbb{R}^d) \Rightarrow \hat{f} * \hat{g} \in L^1(\mathbb{R}^d)$ a

$$\widehat{\hat{f} * \hat{g}} = \hat{\hat{f}} \cdot \hat{\hat{g}} = \check{f} \cdot \check{g} = \widecheck{f \cdot g} = \widehat{\widehat{f \cdot g}}$$

$$\Rightarrow \hat{f} * \hat{g} = \widehat{f \cdot g}$$

Tedy $\mathcal{S}$ je uzavřený na konvoluci:

$$f, g \in \mathcal{S} \Rightarrow f * g = \hat{\check{f}} * \hat{\check{g}} = \widehat{\check{f} \cdot \check{g}} \in \mathcal{S}$$

Věta 34

① $f \in \mathcal{S} \Rightarrow \|f\|_2 = \|\hat{f}\|_2$

$$\|f\|_2^2 = \int_{\mathbb{R}^d} f \cdot \bar{f} \, dm_d = \int_{\mathbb{R}^d} f \cdot \hat{\check{\bar{f}}} \, dm_d = \int_{\mathbb{R}^d} \hat{f} \, \check{\bar{f}} \, dm_d = \|\hat{f}\|_2^2,$$

protože $\check{\bar{f}}(t) = \int_{\mathbb{R}^d} \bar{f}(x) e^{-i\langle t, x\rangle} dm_d(x) = \int_{\mathbb{R}^d} \overline{f(-x)} \, e^{-i\langle t, x\rangle} dm_d(x)$

$$= \overline{\int_{\mathbb{R}^d} f(-x) \, e^{-i\langle t, -x\rangle} dm_d(x)} = \bar{\hat{f}}(t).$$

② $\mathcal{S}$ je hustý podprostor $L^2(\mathbb{R}^d)$ (protože $\mathcal{S} \supset \mathcal{D}(\mathbb{R}^d)$),

tedy rozšíření ex. z Věty I. 15

③

(3) $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \Rightarrow \mathcal{P}(f) = \mathcal{F}(f) \ (= \hat{f})$

- Nech $r > 0$ a $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $f = 0$ na $\mathbb{R}^d \setminus B(0,r)$

  $\Rightarrow$ vlastne $f \in L^2(B(0,r))$

  $\Rightarrow$ ex. $\varphi_n \in \mathcal{D}(U(0,r))$: $\varphi_n \to f$ v L^2

  Pretože $U(0,r)$ má konečnú mieru, je $\varphi_n \to f$ v L^1

  Pretože $\varphi_n \in \mathcal{D}(U(0,r)) \subset \mathcal{S}(\mathbb{R}^d)$, je $\mathcal{P}(\varphi_n) = \hat{\varphi}_n$

  Teď $\hat{\varphi}_n \rightrightarrows \hat{f}$ (konvergencia v $C_0(\mathbb{R})$)

  a $\hat{\varphi}_n = \mathcal{P}(\varphi_n) \to \mathcal{P}(f)$ v L^2

  ex. podpost., ktorá konverguje s.v., teď $\mathcal{P}(f) = \hat{f}$ s.v.

  $\Rightarrow \mathcal{P}(f) = \hat{f}$ v $L^1 \cap L^2$

- $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ obecná, označíme $f_n := f \cdot \chi_{B(0,n)}$

  Dle predchozího bodu je $\mathcal{P}(f_n) = \hat{f}_n$ pre každé $n \in \mathbb{N}$

  Máme $f_n \to f$ v $L^2 \Rightarrow \mathcal{P}(f_n) \to \mathcal{P}(f)$ v L^2

  $f_n \to f$ v $L^1 \Rightarrow \hat{f}_n \rightrightarrows \hat{f}$

  Opäť ex. podpost. $\mathcal{P}(f_n) = \hat{f}_n$, ktorá konverguje s.v. k $\mathcal{P}(f)$, teď $\mathcal{P}(f) = \hat{f}$ s.v., tj. $\mathcal{P}(f) = \hat{f}$ v $L^1 \cap L^2$.