Lemma 42: $\varphi \in \mathcal{S}(\mathbb{R}^d)$

(a) $x_n \to x$ v $\mathbb{R}^d \Rightarrow \tau_{x_n}\varphi \to \tau_x \varphi$ v $\mathcal{S}(\mathbb{R}^d)$

⌈ $N \in \mathbb{N}_0$, α multiindeks, $|\alpha| \le N$

$$\cancel{(1+\|y\|^2)^N D_\alpha (\tau_{x_n}\varphi - \tau_x\varphi(y))}$$

Zvolimo $x \in \mathbb{R}^d$ poljuben a $z \in \mathbb{R}^d$, $\|x-z\| < \frac{1}{2}$

Poglejmo (za $y \in \mathbb{R}^d$) $\left|(1+\|y\|^2)^N D^\alpha (\tau_z\varphi - \tau_x\varphi)(y)\right| =$

$$= \left|(1+\|y\|^2)^N \left(D^\alpha\varphi(y-z) - D^\alpha\varphi(y-x)\right)\right| =$$

$$= (1+\|y\|^2)^N \left| \int_0^1 \frac{\partial}{\partial t} D^\alpha \varphi\big(y - (x + t(z-x))\big)\, dt \right| =$$

$$= (1+\|y\|^2)^N \left| \int_0^1 \sum_{j=1}^d \frac{\partial}{\partial y_j} D^\alpha \varphi\big(y - x - t(z-x)\big)\cdot (x_j - z_j)\, dt \right|$$

$$\le (1+\|y\|^2)^N \int_0^1 \sum_{j=1}^d \frac{p_{N+1}(\varphi)}{(1+\|y-x-t(z-x)\|^2)^{N+1}} |x_j - z_j|\, dt \le$$

$$\le (1+\|y\|^2)^N \cdot p_{N+1}(\varphi) \cdot \int_0^1 \left(\sum_{j=1}^d \frac{1}{(1+\|y-x-t(z-x)\|^2)^{2(N+1)}}\right)^{1/2} \cdot \|x-z\|\, dt$$

$$\le p_{N+1}(\varphi) \cdot \|x-z\| \cdot (1+\|y\|^2)^N \cdot \int_0^1 \frac{\sqrt{d}\; dt}{(1+(\|y-x\| - \|t(z-x)\|)^2)^{N+1}}$$

$$\le p_{N+1}(\varphi)\, \|x-z\|\, (1+\|y\|^2)^N \sqrt{d} \int_0^1 \frac{dt}{\left(1+\|y-x\|^2 - 2\|y-x\|\,\|t(z-x)\| + \|t(z-x)\|^2\right)^{N+1}}$$

$$\le p_{N+1}(\varphi) \cdot \|x-z\| \cdot \sqrt{d} \cdot \frac{(1+\|y\|^2)^N}{(1 - \|y-x\| + \|y-x\|^2)^{N+1}}$$

⊛ $p_N(\tau_z\varphi - \tau_x\varphi) \to 0$

za $z \to x$

to je spojita funkcija na $\mathbb{R}^d$, v okolici 0

$\Rightarrow$ je omejena. Odtod pa sledi ⊛ ⌋

(3) $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $e \in \mathbb{R}^d$ $\Rightarrow$ $\varphi_r(x) = \dfrac{\varphi(x+re)-\varphi(x)}{r}$ sledi $\varphi_r \to \partial_e \varphi$ v $\mathcal{S}(\mathbb{R}^d)$ ko $r \to 0$

Dk: $N \in \mathbb{N}_0$, α multiindeks, $|\alpha| \le N$

Naj bo $r \in \mathbb{R}\setminus\{0\}$, $\|re\| < \frac{1}{2}$. Pač za $x \in \mathbb{R}^d$:

$$\left|(1+\|x\|^2)^N D^\alpha(\varphi_r - \partial_e\varphi)(x)\right| = \left|(1+\|x\|^2)^N\left(\frac{1}{r}\left(D^\alpha\varphi(x+re) - D^\alpha\varphi(x)\right) - \partial_e D^\alpha\varphi(x)\right)\right|$$

$$= \left|(1+\|x\|^2)^N \left(\frac{1}{r}\int_0^r \left(\partial_e \varphi(x+te) - \partial_e\varphi(x)\right) dt\right)\right| =$$

$$= \left|(1+\|x\|^2)^N \frac{1}{r}\int_0^r\int_0^t \partial_e\partial_e D^\alpha\varphi(x+se)\, ds\, dt\right| =$$

$$= \left|(1+\|x\|^2)^N \frac{1}{r}\int_0^r\int_0^t \sum_{k,l=1}^d \frac{\partial^2}{\partial x_k \partial x_l} D^\alpha\varphi(x+se)\, e_k e_l\, ds\, dt\right| \le$$

$$\le (1+\|x\|^2)^N \frac{1}{|r|}\int_{[0,r]}\int_{[0,t]} \frac{P_{N+2}(\varphi)}{(1+\|x+se\|^2)^{N+2}} \cdot \|e\|_1^2\, ds\, dt \le$$

$$\le \frac{(1+\|x\|^2)^N}{(1-\|x\|+\|x\|^2)^{N+2}} \cdot \|e\|_1^2 \cdot P_{N+2}(\varphi) \cdot \frac{1}{|r|} \underbrace{\int_{[0,r]}\int_{[0,t]} 1\, ds\, dt}_{=\frac{1}{2}|r|^2}$$

podobno kot v (c)

$$= \underbrace{\frac{(1+\|x\|^2)^N}{(1-\|x\|+\|x\|^2)^{N+2}}}_{\text{omejeno na } \mathbb{R}^d} \cdot \|e\|_1^2\, P_{N+2}(\varphi) \cdot \frac{1}{2}|r|$$

$\Rightarrow P_N(\varphi_r - \partial_e\varphi) \to 0$ ko $r \to 0$.

Tvrzení 43(a) $\varphi \in \mathcal{S}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$, $\Lambda \in \mathcal{S}'(\mathbb{R}^{d_1})$

Pak $\quad \psi(y) = \Lambda(x \mapsto \varphi(x,y))$ patří do $\mathcal{S}(\mathbb{R}^{d_2})$

a $\quad D^\alpha \psi(y) = \Lambda(x \mapsto D^{(0,\alpha)}\varphi(x,y))$

- zřejmě $x \mapsto \varphi(x,y)$ (kde y pevné) patří do $\mathcal{S}(\mathbb{R}^{d_1})$, tedy ψ je dobře definována!

- Ukážeme, že ψ je spojitá:

$$y \in \mathbb{R}^{d_2} \quad \dots \quad \varphi^{y}(x) = \varphi(x,y)$$

Tvrdím, že $\quad y_k \to y \Rightarrow \varphi^{y_k} \to \varphi^{y}$ v $\mathcal{S}(\mathbb{R}^{d_1})$

$N \in \mathbb{N}_0$, α multiindex, $|\alpha| \leq N$

$$|(1+\|x\|^2)^N D^\alpha(\varphi^{y_k} - \varphi^{y})(x)| = |(1+\|x\|^2)^N (D^{(\alpha,0)}\varphi(x,y_k) - D^{(\alpha,0)}\varphi(x,y))|$$

$$= (1+\|x\|^2)^N \left| \int_0^1 \frac{\partial}{\partial t} D^{(\alpha,0)} \varphi(x, y + t(y_k - y))\, dt \right|$$

$$\leq (1+\|x\|^2)^N \left| \int_0^1 \sum_{j=1}^{d} \frac{\partial}{\partial y_j} D^{(\alpha,0)} \varphi(x, y + t(y_k - y))(y_{k,j} - y_j)\, dt \right|$$

$$\leq (1+\|x\|^2)^N \int_0^1 \sum_{j=1}^{d} \frac{P_{N+1}(\varphi)}{(1+\|x\|^2 + \|y + t(y_k - y)\|^2)^{N+1}} |y_{k,j} - y_j|\, dt$$

$$\leq \frac{P_{N+1}(\varphi)}{(1+\|x\|^2)} \cdot \|y_k - y\|_1 \leq P_{N+1}(\varphi) \|y_k - y\|_1$$

$$\Rightarrow P_N(\varphi^{y_k} - \varphi^{y}) \to 0.$$

Máme tedy $\varphi^{y_k} \to \varphi^{y}$ v $\mathcal{S}(\mathbb{R}^{d})$, tedy $\Lambda(\varphi^{y_k}) \to \Lambda(\varphi^{y})$,

neboli $\psi(y_k) \to \psi(y)$

Dále ukážeme, že $D^\alpha \psi(y) = \Lambda(x \mapsto D^{(0,\alpha)} \varphi(x,y))$

Stačí pro $|\alpha|=1$, další postupně a z předchozího plyne, že $\psi \in C^\infty$

$$\frac{\partial}{\partial y_1}\psi(y) = \lim_{t\to 0} \frac{\psi(y+te_1)-\psi(y)}{t} = \lim_{t\to 0} \Lambda\left(x \mapsto \frac{\varphi(x,y+te_1)-\varphi(x,y)}{t}\right)$$

$$= \Lambda\left(x \mapsto \frac{\partial}{\partial y_1}\varphi(x,y)\right)$$

Lemma 42(5)

(protože $\left(x \mapsto \frac{\varphi(x,y+te_1)-\varphi(x,y)}{t}\right) \longrightarrow \left(x \mapsto \frac{\partial}{\partial y_1}\varphi(x,y)\right)$ v $\mathcal{S}(\mathbb{R}^{d_1})$

[Lemma 42 dává konvergenci v $\mathcal{S}(\mathbb{R}^{d_1+d_2})$, což je silnější]

nyní: $N \in \mathbb{N}_0$, $|\alpha| \le N$, pak

$$\left|(1+\|y\|^2)^N D^\alpha \psi(y)\right| = \left|\Lambda\left(x \mapsto (1+\|y\|^2)^N D^{(0,\alpha)}\varphi(x,y)\right)\right|$$

$$\le C \cdot p_M\left(x \mapsto (1+\|y\|^2)^N D^{(0,\alpha)}\varphi(x,y)\right)$$

a. $M \in \mathbb{N}_0$, $C>0$ z T 36(a)

$$\le C \cdot p_{M+N}(\varphi)$$

Tedy $p_N(\psi) \le C \cdot p_{M+N}(\varphi)$.

<u>Tvrzení</u>: 43(b) $\varphi \in \mathcal{S}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$, $\Lambda_1 \in \mathcal{S}'(\mathbb{R}^{d_1})$, $\Lambda_2 \in \mathcal{S}'(\mathbb{R}^{d_2})$

Pak $\Lambda_1\big(x \mapsto \Lambda_2(y \mapsto \varphi(x,y))\big) = \Lambda_2\big(y \mapsto \Lambda_1(x \mapsto \varphi(x,y))\big)$

<u>Dk:</u> dle předchozího tvrzení (a) mají obě strany smysl.

Použijeme <u>Lemma $\tilde{I}$</u>: $\Lambda \in \mathcal{S}'(\mathbb{R}^d) \Rightarrow \exists\, N \in \mathbb{N}_0$ a (konečné) míry μ_α, $|\alpha| \le N$ na $\mathbb{R}^d$, že

$$\Lambda(\varphi) = \sum_{|\alpha| \le N} \int_{\mathbb{R}^d} (1+|x|^2)^N D^\alpha \varphi(x)\, d\mu_\alpha(x)$$

⌈ dle T 36 (a) existuje $N \in \mathbb{N}_0$ a $C > 0$, že

$$|\Lambda(\varphi)| \le C \cdot p_N(\varphi), \quad \varphi \in \mathcal{S}$$

Uvažme $(\mathcal{S}, p_N)$. To je NLP a Λ je spojitý lin. funkcionál na $(\mathcal{S}, p_N)$.

Definujme zobrazení $T: \mathcal{S} \to \prod_{|\alpha| \le N} C_0(\mathbb{R})$ předpisem

$$T\varphi(x) = \left((1+|x|^2)^N D^\alpha \varphi(x) \right)_{|\alpha| \le N}$$

Pak na $X := \prod_{|\alpha| \le N} C_0(\mathbb{R})$ definujeme normu

$\|(f_\alpha)_{|\alpha| \le N}\| = \max_{|\alpha| \le N} \|f_\alpha\|_\infty$, je T izometrie $(\mathcal{S}, p_N)$ do X

Tedy $\tilde{\Lambda} := \Lambda \circ T^{-1}$ je spojitý lin. funkcionál na $T(\mathcal{S})$, $\|\tilde{\Lambda}\| \le C$ $\Rightarrow$ lze rozšířit se zachováním normy na $\Phi \in X^*$

$\Phi \in \left(\prod_{|\alpha| \leq N} C_0(\mathbb{R}^d) \right)^* \Rightarrow$ z Riesza wiąż o reprezentacji $C_0(\mathbb{R}^d)^*$

istnieją $(\mu_\alpha)_{|\alpha| \leq N}$, że $\Phi((f_\alpha)_{|\alpha| \leq N}) = \sum_{|\alpha| \leq N} \int_{\mathbb{R}^d} f_\alpha \, d\mu_\alpha$

Para $\Lambda(\varphi) = \Phi(T\varphi) = \sum_{|\alpha| \leq N} \int_{\mathbb{R}^d} (1+\|t\|^2)^N D^\alpha \varphi(t) \, d\mu_\alpha(t)$ $\lrcorner$

$\Lambda_1 \ldots N_1, \mu_\alpha, |\alpha| \leq N_1, \quad \Lambda_2 \ldots N_2, \nu_\alpha, |\alpha| \leq N_2$

$$\Lambda_1(x \mapsto \Lambda_2(y \mapsto \varphi(x,y))) = \Lambda_1\Big(t \mapsto \sum_{|\beta| \leq N_2} \int_{\mathbb{R}^{d_2}} (1+\|y\|^2)^{N_2} D^{(0,\beta)} \varphi(t,y) \, d\nu_\beta(y)\Big)$$

$$= \sum_{|\alpha| \leq N_1} \int_{\mathbb{R}^{d_1}} (1+\|t\|^2)^{N_1} D^\alpha \Big(\sum_{|\beta| \leq N_2} \int_{\mathbb{R}^{d_2}} (1+\|y\|^2)^{N_2} D^{(0,\beta)} \varphi(t,y) \, d\nu_\beta(y) \Big) d\mu_\alpha(t)$$

$$= \sum_{|\alpha| \leq N_1} \sum_{|\beta| \leq N_2} \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} (1+\|t\|^2)^{N_1} (1+\|y\|^2)^{N_2} D^{(\alpha,\beta)} \varphi(t,y) \, d\nu_\beta(y) \, d\mu_\alpha(t)$$

$$\overset{\text{Fubini}}{=} \sum_{|\alpha| \leq N_1} \sum_{|\beta| \leq N_2} \int_{\mathbb{R}^{d_2}} (1+\|y\|^2)^{N_2} D^\beta \Big(\int_{\mathbb{R}^{d_1}} (1+\|t\|^2)^{N_1} D^{(\alpha,0)} \varphi(t,y) \, d\mu_\alpha(t) \Big) d\nu_\beta(y)$$

$$= \Lambda_2(y \mapsto \Lambda_1(x \mapsto \varphi(x,y)))$$