V44 $U \in \mathcal{S}'$, $\varphi, \psi \in \mathcal{S}$

(a) $U * \varphi \in C^\infty(\mathbb{R}^d)$, $D^\alpha(U*\varphi) = (D^\alpha U)*\varphi = U * D^\alpha \varphi$

⌈ $U * D^\alpha\varphi(t) = U(y \mapsto D^\alpha \varphi(t-y))$

$D^\alpha U * \varphi(t) = D^\alpha U(y \mapsto \varphi(t_0 - y)) = (-1)^{|\alpha|} U(D^\alpha(y\mapsto \varphi(t-y))) =$

$= (-1)^{|\alpha|} U(y \mapsto (-1)^{|\alpha|} D^\alpha\varphi(t-y)) = U(y\mapsto D^\alpha \varphi(t-y))$

$\Rightarrow U * D^\alpha\varphi = D^\alpha U * \varphi$

$U*\varphi$ je spojitá: $t_n \to x \Rightarrow \tau_{t_n}\check\varphi \to \tau_x \check\varphi$ v $\mathcal{S}$ (Lemma)

$\Rightarrow U(\tau_{t_n}\check\varphi) \to U(\tau_x\check\varphi) \Rightarrow U*\varphi(t_n) \to U*\varphi(t)$

$$\frac{\partial}{\partial x_1} U*\varphi(t) = \lim_{t\to0} \frac{U*\varphi(t+te_1) - U*\varphi(t)}{t} = \lim_{t\to 0} U\Big(y \mapsto \underbrace{\frac{\varphi(t+te_1-y)-\varphi(t-y)}{t}}\Big)$$

↓ Lemma 42(b)

$y \mapsto \frac{\partial}{\partial t_1}\varphi(t-y)$ v $\mathcal S$

$= U\left(y\mapsto \frac{\partial}{\partial t_1}\varphi(t-y)\right)$

$= U * \frac{\partial}{\partial t_1}\varphi(t)$ a dále indukcí ⌋

(b) $\Lambda_{U*\varphi}$ je temperovaná distribuce

⌈ $U\in\mathcal S' \Rightarrow \exists\, C>0, N\in\mathbb N_0$, že $|U(g)| \le C p_N(g)$ pro $g \in \mathcal S$

Pak $|U*\varphi(t)| = |U(\tau_x\check\varphi)| \le C\cdot p_N(\tau_t\check\varphi)$

Odhadneme $p_N(\tau_x\check\varphi)$

$|\alpha| \le N,\ y \in \mathbb{R}^d$: $\quad |(1+\|y\|^2)^N D^\alpha(y \mapsto \varphi(x-y))| =$

$$= (1+\|y\|^2)^N |D^\alpha \varphi(x-y)| \le \frac{(1+\|y\|^2)^N}{(1+\|x-y\|^2)^N} \cdot p_N(\varphi)$$

Pri tem $\dfrac{1+\|y\|^2}{1+\|x-y\|^2} = \dfrac{1+\|x-y\|^2 + 2\langle y-x, x\rangle + \|x\|^2}{1+\|x-y\|^2} =$

$$= 1 + \frac{2\langle y-x, x\rangle}{1+\|x-y\|^2} + \frac{\|x\|^2}{1+\|x-y\|^2} \le 1 + \frac{2\|y-x\|\,\|x\|}{1+\|x-y\|^2} + \|x\|^2$$

$$\le 1 + \|x\| + \|x\|^2$$

$$\Rightarrow p_N(\tau_x \check{\varphi}) \le (1+\|x\|+\|x\|^2)^N \cdot p_N(\varphi)$$

$$\Rightarrow |U * \varphi(x)| \le (1+\|x\|+\|x\|^2)^N \cdot p_N(\varphi)$$

$\Rightarrow \Lambda_{U*\varphi} \in \mathcal{S}'$ dle T37(c)

(c) $f \in L^p(\mathbb{R}^d) \Rightarrow \Lambda_f * \varphi = f * \varphi$

$$\Lambda_f * \varphi(x) = \Lambda_f(y \mapsto \varphi(x-y)) = \int_{\mathbb{R}^d} f(y)\varphi(x-y)\,dm_d(y) = f * \varphi(x)$$

dy ~~[illegible]~~ (y)

a) $\widehat{\Lambda_{U*\varphi}} = \hat{\varphi}\cdot\hat{U}$

$$\widehat{\Lambda_{U*\varphi}}(\psi) = \Lambda_{U*\varphi}(\hat{\psi}) = \int_{\mathbb{R}^d} U*\varphi(t)\cdot\hat{\psi}(t)\,dm_d(t)$$

$$= \int_{\mathbb{R}^d} U(y\mapsto\varphi(t-y))\cdot\hat{\psi}(t)\,dm_d(t) =$$

$$= \int_{\mathbb{R}^d} U(y\mapsto\varphi(t-y)\hat{\psi}(t))\,dm_d(t) =$$

$$= \Lambda_1(t\mapsto U(y\mapsto\varphi(t-y)\hat{\psi}(t))) \overset{\substack{\text{T4.3(5)}\\(*)}}{=} U(y\mapsto\Lambda_1(t\mapsto\varphi(t-y)\hat{\psi}(t))) =$$

$$= U\left(y\mapsto\int_{\mathbb{R}^d}\tau_y\varphi\cdot\hat{\psi}\,dm_d\right) = U\left(y\mapsto\int_{\mathbb{R}^d}\widehat{\tau_y\varphi}\cdot\psi\,dm_d\right) =$$

$$= U\left(y\mapsto\int_{\mathbb{R}^d} e_{-y}\hat{\varphi}\psi\right) = U(\widehat{\hat{\varphi}\cdot\psi}) = \hat{U}(\hat{\varphi}\cdot\psi) =$$

$$= \hat{\varphi}\cdot\hat{U}(\psi)$$

(*) $(t,y)\mapsto\varphi(t-y)\psi(t)$ patří do $\mathcal{S}(\mathbb{R}^{d+d})$ pro každá $\varphi,\psi\in\mathcal{S}(\mathbb{R}^d)$

- je to C^∞
- $\left|(1+\|t\|^2+\|y\|^2)^N D^{(\alpha,\beta)}(\varphi(t-y)\psi(t))\right| =$

$$= \left|(1+\|t\|^2+\|y\|^2)^N \sum_{\gamma\le\alpha} c_{\gamma,\alpha}(-1)^{|\beta|} D^{\gamma+\beta}\varphi(t-y)\, D^{\alpha-\gamma}\psi(t)\right|$$

$$\le (1+\|t\|^2+\|y\|^2)^N \sum_{\gamma\le\alpha} c_{\gamma,\alpha}\,\frac{p_N(\varphi)\qquad p_N(\psi)}{(1+\|t-y\|^2)^N\,(1+\|t\|^2)^N}$$

$$\frac{1+\|t\|^2+\|y\|^2}{(1+\|t-y\|^2)(1+\|t\|^2)} \begin{cases} \overset{\|y\|\le 2\|t\|}{\le} \dfrac{1+5\|t\|^2}{1+\|t\|^2} \le 5 \\ \|y\|>2\|t\| \Rightarrow \|t-y\|\ge\|y\|-\|t\|>\dfrac{\|y\|}{2}\end{cases}$$

$$\text{tez} \le \frac{1+5\|y\|^2}{1+\|\frac{y}{2}\|^2} \le \frac{5}{\frac{1}{4}} = 20$$

$$\widehat{\varphi \cdot \upsilon} = 1_{\hat{\varphi} * \hat{\upsilon}}$$

$$1_{\hat{\varphi} * \hat{\upsilon}} = \widehat{\widehat{\widehat{\widehat{1_{\hat{\varphi} * \hat{\upsilon}}}}}} = \widehat{\widehat{\widehat{\hat{\varphi} \cdot \hat{\upsilon}}}} = \widehat{\widehat{\widehat{\check{\varphi} \cdot \check{\upsilon}}}} =$$

$$= \widehat{\widehat{\check{\varphi} \cdot \check{\upsilon}}} = \widehat{\check{\check{\varphi}} \cdot \check{\check{\upsilon}}} = \widehat{\varphi \cdot \upsilon}$$

(e) $U * (\varphi * \psi) = (U * \varphi) * \psi$

$$U * (\varphi * \psi)(x) = U(y \mapsto \varphi * \psi(x-y)) = U\Big(y \mapsto \int_{\mathbb{R}^d} \varphi(x-y-z)\psi(z)\, d\lambda_d(z)\Big)$$

$$\overset{T43(3)}{\underset{(*)}{=}} U\big(y \mapsto \Lambda_1(z \mapsto \varphi(x-y-z)\psi(z))\big) = \Lambda_1\big(z \mapsto U(y \mapsto \varphi(x-y-z)\psi(z))\big)$$

$$= \Lambda_1\big(z \mapsto \psi(z)\, U(y \mapsto \varphi(x-y-z))\big) = \Lambda_1\big(z \mapsto \psi(z) \cdot U * \varphi(x-z)\big) =$$

$$= (U * \varphi) * \psi(x)$$

(*) Funk. $(y,z) \mapsto \varphi(x-y-z)\psi(z)$ pripada $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$

- je C^∞
- $N \in \mathbb{N}_0$, $|\beta|+|\alpha| \le N$, $(y,z) \in \mathbb{R}^d$:

$$\left|(1+\|y\|^2+\|z\|^2)^N D^{(\alpha,\beta)}\big(\varphi(x-y-z)\psi(z)\big)\right| = (1+\|y\|^2+\|z\|^2)^N \cdot$$

$$\cdot \left|\sum_{\gamma \le \beta} (-1)^{|\alpha|+|\beta|} c_{\gamma,\beta}\, D^{\alpha+\gamma}\varphi(x-y-z)\, D^{\beta-\gamma}\psi(z)\right| \le$$

$$\le (1+\|y\|^2+\|z\|^2)^N \cdot \sum_{\gamma \le \beta} c_{\gamma,\beta} \cdot \frac{p_N(\varphi)}{(1+\|x-y-z\|^2)^N} \cdot \frac{p_N(\psi)}{(1+\|z\|^2)^N}$$

Prištam: $$\frac{1+\|y\|^2+\|z\|^2}{(1+\|x-y-z\|^2)(1+\|z\|^2)} = \underbrace{\frac{1}{1+\|x-y-z\|^2}}_{\le 1} + \frac{\|y\|^2}{(1+\|x-y-z\|^2)(1+\|z\|^2)}$$

$$\frac{\|y\|^2}{(1+\|x-y-z\|^2)(1+\|z\|^2)}$$

- $\|y\| \le 2\|z\|$: ≤ 4
- $\|y\| > 2\|z\|$, $\|y+z\| > 2\|x\|$ $\Rightarrow \|x-y-z\| > \frac{1}{2}\|y+z\| > \frac{1}{4}\|y\|$ $\Rightarrow$ " ≤ 16 "
- $\|y\| > 2\|z\|$, $\|y+z\| \le 2\|x\|$ $\Rightarrow \|y+z\| \ge \frac{1}{2}\|y\| \Rightarrow \|y\| \le 4\|x\| \Rightarrow$ " $\le 16\|x\|^2$ "