Nechť $p \in (1,\infty)$, $p \neq 2$. Uvažme $(\mathbb{R}^2, \|\cdot\|_p)_C$:

$$\|(x,0),(0,y)\|_{min} = \sup\{\|\alpha(x,0)+\beta(0,y)\|_p : \alpha^2+\beta^2=1\} =$$
$$= \sup\{\|(\alpha x,\beta y)\|_p : \alpha^2+\beta^2=1\} = \sup\{(|\alpha x|^p+|\beta y|^p)^{1/p} : \alpha^2+\beta^2=1\}$$

To je úloha na vázaný extrém : $f(\alpha,\beta) = |\alpha x|^p + |\beta y|^p$

na $\{(\alpha,\beta) : \alpha^2+\beta^2=1\}$

$(g(\alpha,\beta) = \alpha^2+\beta^2-1)$

- $f \in C^\infty(\mathbb{R}^2 \setminus \{(\alpha,\beta) : \alpha=0 \text{ nebo } \beta=0\})$

$\alpha=0 \Rightarrow \beta=\pm 1$, pak $f(0,\pm 1) = |y|^p$

$\beta=0 \Rightarrow \alpha=\pm 1$, pak $f(\pm 1,0) = |x|^p$

V ostatních bodech, pokud extrém, pak $\nabla f = \lambda\cdot\nabla g$

$\nabla g = (2\alpha, 2\beta)$, $\nabla f = (|x|^p\cdot p\cdot|\alpha|^{p-1}\cdot\operatorname{sgn}\alpha,\ |y|^p\cdot p\cdot|\beta|^{p-1}\cdot\operatorname{sgn}\beta)$

$$|x|^p\cdot p\cdot|\alpha|^{p-1}\operatorname{sgn}\alpha = \lambda\cdot 2\alpha \quad / \ \beta\cdot\text{první} - \alpha\cdot\text{druhý}$$
$$|y|^p\cdot p\cdot|\beta|^{p-1}\cdot\operatorname{sgn}\beta = \lambda\cdot 2\beta$$

- $|x|^p\cdot p\cdot|\alpha|^{p-1}\operatorname{sgn}\alpha\cdot\beta = |y|^p\cdot p\cdot|\beta|^{p-1}\operatorname{sgn}\beta\cdot\alpha \quad / :(p\cdot\alpha\cdot\beta)$

$$|x|^p\cdot|\alpha|^{p-2} = |y|^p\cdot|\beta|^{p-2}$$

pokud $x=0$ nebo $y=0$, pak odsud $x=y=0$ (protože $\alpha\neq 0\neq\beta$)
a v bodě, kde $x=y=0$ je f konstantní 0.

Tedy $|\beta| = |\alpha|\cdot\left|\frac{x}{y}\right|^{\frac{p}{p-2}}$. A pak dosadíme do vazby:

$$\alpha^2\left(1+\left|\frac{x}{y}\right|^{\frac{2p}{p-2}}\right) = 1$$

$$\alpha^2 = \frac{|y|^{\frac{2p}{p-2}}}{\left(|x|^{\frac{2p}{p-2}}+|y|^{\frac{2p}{p-2}}\right)} \Rightarrow |\alpha| = \frac{|y|^{\frac{p}{p-2}}}{\left(|x|^{\frac{2p}{p-2}}+|y|^{\frac{2p}{p-2}}\right)^{1/2}}$$

Tedy $|\beta| = \dfrac{|x|^{\frac{p}{p-2}}}{\left(|x|^{\frac{2p}{p-2}}+|y|^{\frac{2p}{p-2}}\right)^{1/2}}$

$$f(\ , \ ) = \frac{|y|^{\frac{p^2}{p-2}} \cdot |x|^p + |x|^{\frac{p^2}{p-2}} \cdot |y|^p}{\left(|x|^{\frac{2p}{p-2}} + |y|^{\frac{2p}{p-2}}\right)^{\frac{p}{2}}} =$$

$$= |x|^p |y|^p \cdot \frac{|y|^{\frac{2p}{p-2}} + |x|^{\frac{2p}{p-2}}}{\left(|x|^{\frac{2p}{p-2}} + |y|^{\frac{2p}{p-2}}\right)^{\frac{p}{2}}} = |x|^p |y|^p \cdot \left(|x|^{\frac{2p}{p-2}} + |y|^{\frac{2p}{p-2}}\right)^{1-\frac{p}{2}}$$

Závěr:

$$\|(x,0),(0,y)\|_{min} = \max\left\{|x|, |y|, |x||y| \cdot \left(|x|^{\frac{2p}{p-2}} + |y|^{\frac{2p}{p-2}}\right)^{\frac{1}{p}-\frac{1}{2}}\right\}$$

Přitom $\|(x,0)+i\cdot(0,y)\|_p = \|(x,iy)\|_p = \left(|x|^p + |y|^p\right)^{1/p}$

Například: $x = y = 1$: $\|(1,0),(0,1)\|_{min} = \max\left\{1, 2^{\frac{1}{p}-\frac{1}{2}}\right\}$

$\|(1,0)+i(0,1)\|_p = 2^{1/p}$

Případ $p=2$: vypočítal jsem stejně ..... $\alpha = 0, \beta = \pm 1 \Rightarrow f(0,\pm 1) = y^2$

$\alpha = \pm 1 \ \beta = 0 \Rightarrow f(\pm 1, 0) = x^2$

$\alpha \neq 0 \neq \beta$ ---> vyjde $x^2 = y^2$

tedy, pokud $x^2 \neq y^2$, není řešení pro α, β

pokud $x^2 = y^2$, pak vyhovují všechna α, β

$\Rightarrow$ pro $x^2 \neq y^2$ je $\|(x,0),(0,y)\|_{min} = \max\{|x|, |y|\}$

pro $x^2 = y^2$ je $f(\alpha,\beta) = |\alpha x|^2 + |\beta y|^2 = \alpha^2 x^2 + \beta^2 x^2 = x^2$

$\Rightarrow$ je to každé!

Závěr: $\|(x,0),(0,y)\|_{min} = \max\{|x|, |y|\}$

$\|(x,0)+i(0,y)\|_2 = (x^2+y^2)^{1/2}$

} to je evidentně různo!