Domače TP 1 ($f,g \in L^1(\mathbb{R}^d)$, $t,y \in \mathbb{R}^d$)

(a) $\widehat{\tau_y f} = e_{-y} \cdot \hat{f}$

$$\widehat{\tau_y f}(t) = \int_{\mathbb{R}^d} \tau_y f(x)\, e^{-i\langle t,x\rangle}\, dm_d(x) =$$

$$= \int_{\mathbb{R}^d} f(x-y)\, e^{-i\langle t,x-y\rangle} \cdot e^{-i\langle t,y\rangle}\, dm_d(x)$$

$$= e_{-y}(t) \cdot \hat{f}(t)$$

(b) $\widehat{e_y \cdot f} = \tau_y \hat{f}$

$$\widehat{e_y \cdot f}(t) = \int_{\mathbb{R}^d} e_y(x) \cdot f(x) \cdot e^{-i\langle t,x\rangle}\, dm_d(x) =$$

$$= \int_{\mathbb{R}^d} f(x)\, e^{-i\langle t-y,x\rangle}\, dm_d(x) = \hat{f}(t-y) = \tau_y \hat{f}(t)$$

(c) $\widehat{f*g} = \hat{f} \cdot \hat{g}$

$\ulcorner f, g \in \mathcal{L}^1(\mathbb{R}^d) \Rightarrow$ vieme, že $f*g \in \mathcal{L}^1(\mathbb{R}^d)$

$$\widehat{f*g}(t) = \int_{\mathbb{R}^d} f*g(x)\, e^{-i\langle t, x\rangle}\, dm_d(x) =$$

$$= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x-y)\, g(y)\, dm_d(y) \right) e^{-i\langle t, x\rangle}\, dm_d(x)$$

$$= \int_{\mathbb{R}^d} \Big(g(y)\, e^{-i\langle t, y\rangle} \cdot \underbrace{\int_{\mathbb{R}^d} f(x-y)\, e^{-i\langle t, x-y\rangle}\, dm_d(x)}_{\hat{f}(t)} \Big)\, dm_d(y) =$$

$$= \hat{f}(t) \cdot \hat{g}(t)$$

Fubiniova veta (funkcia $(x,y) \mapsto f(x-y)\, g(y)\, e^{-i\langle t, x\rangle}$ je integrovateľná dĺž F.V. pre nezáp. funkc) $\lrcorner$

(d) $\lambda > 0$, $h(x) = f\left(\frac{x}{\lambda}\right) \Rightarrow \hat{h}(t) = \lambda^d \hat{f}(\lambda t)$

$$\ulcorner \hat{h}(t) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f\left(\frac{x}{\lambda}\right) e^{-i\langle t, x\rangle}\, dx \underset{\substack{x = \lambda \cdot y \\ \text{jakobián} = \lambda^d}}{=}$$

$$= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(y)\, e^{-i\langle t, \lambda y\rangle} \lambda^d\, dy =$$

$$= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(y)\, e^{-i\langle \lambda t, y\rangle} \lambda^d\, dy = \lambda^d \hat{f}(\lambda t) \lrcorner$$

(e) $\int_{\mathbb{R}^d} f(x)\,\hat{g}(x)\,dm_d(x) = \int_{\mathbb{R}^d} f(x) \int_{\mathbb{R}^d} g(y)\, e^{-i\langle x,y\rangle}\, dm_d(y)\, dm_d(x)$

$= \int_{\mathbb{R}^d} g(y) \int_{\mathbb{R}^d} f(x)\, e^{-i\langle x,y\rangle}\, dm_d(x)\, dm_d(y) = \int_{\mathbb{R}^d} g(y)\cdot \hat{f}(y)\, dm_d(y)$

Fubiniova věta

$(x,y) \longmapsto f(x)\,g(y)$ je integrovatelná na $\mathbb{R}^d \times \mathbb{R}^d$)

[Tedy $\int_{\mathbb{R}^d} \hat{f}\, g\, dm_d = \int_{\mathbb{R}^d} f\,\hat{g}\, dm_d$

(f) Nejprve dokažme

Lemma: $h, h' \in L^1(\mathbb{R}) \Rightarrow \lim_{t\to\pm\infty} h(t) = 0$

Dk: $h(t) = h(0) + \int_0^t h' \xrightarrow{t\to+\infty} h(0) + \int_0^\infty h'$ (protože $h' \in L^1$)

pro $t>0$

$\Rightarrow$ limita existuje.

Protože $h \in L^1$, musí být limita 0

Podobně pro $t<0$

$h(t) = h(0) - \int_t^0 h' \xrightarrow{t\to-\infty} h(0) - \int_{-\infty}^0 h'$

$\Rightarrow$ limita existuje, a tedy musí být 0

Nyní: $f, \frac{\partial f}{\partial x_j} \in L^1(\mathbb{R}^d)$. Pro jednoduchost značení předpokládejme $j=1$. Piš $x = (x_1, \tilde{x})$

$\tilde{x} \in \mathbb{R}^{d-1}$ ($\tilde{x} = (x_2, \dots, x_d)$)

Fubiniho věta $\Rightarrow$ pro s.v. $\tilde{x} \in \mathbb{R}^{d-1}$

jsou fce $x_1 \longmapsto f(x_1, \tilde{x})$

$x_1 \longmapsto \frac{\partial f}{\partial x_1}(x_1, \tilde{x})$ v $L^1(\mathbb{R})$

$$\widehat{\frac{\partial f}{\partial x_1}}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_1}(x)\, e^{-i\langle \xi, x\rangle}\, dx =$$

$$= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^{d-1}} \Bigg(\underbrace{\int_{\mathbb{R}} \frac{\partial f}{\partial x_1}(x)\, e^{-i\langle \xi, x\rangle}\, dx_1}_{\Vert \ \text{mo s.v. } \tilde{x}} \Bigg)\, d\tilde{x} = (*)$$

$$\lim_{R \to +\infty} \int_{-R}^{R} \frac{\partial f}{\partial x_1}(x)\, e^{-i\langle \xi, x\rangle} =$$

$$= \lim_{R \to +\infty} \Bigg(\Big[f(x)\, e^{-i\langle \xi, x\rangle} \Big]_{x_1=-R}^{R} -$$

$$- \int_{-R}^{R} f(x) \cdot e^{-i\langle \xi, x\rangle} \cdot (-i\xi_1)\, dx_1 \Bigg)$$

$$\overset{\text{Lemma}}{=} i\xi_1 \int_{\mathbb{R}} f(x)\, e^{-i\langle \xi, x\rangle}\, dx_1$$

$$(*) = \frac{i\xi_1}{(2\pi)^{d/2}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} f(x)\, e^{-i\langle \xi, x\rangle}\, dx_1\, d\tilde{x}$$

$$= \frac{i\xi_1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x)\, e^{-i\langle \xi, x\rangle}\, dx = i\xi_1 \hat{f}(\xi)$$

(g) Nechť fce $g(x)=x_j \cdot f(x)$ patří do $L^1(\mathbb{R}^d)$

$$\hat{f}(t) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \underbrace{f(x)\, e^{-i\langle t,x\rangle}}_{h(t,x)}\, dx$$

- $x \mapsto h(t,x)$ je měřitelná
- $t \mapsto h(t,x)$ je spojitá (pro s.v. x)
- $\frac{\partial}{\partial t} h(t,x) = f(x)\, e^{-i\langle t,x\rangle} \cdot (-i x_j)$

  je spojitá (pro s.v. x)

$$\left|\frac{\partial}{\partial t} h(t,x)\right| = x_j\, |f(x)| = |g(x)|$$

je integrovatelná majoranta

$\Rightarrow$ dle věty o derivaci dle parametru platí:

$$\frac{\partial}{\partial t_j} \hat{f}(t) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} -i x_j\, f(x)\, e^{-i\langle t,x\rangle}\, dx$$

$$\Rightarrow \hat{g} = i \frac{\partial \hat{f}}{\partial t_j}$$

(h) $$\hat{f}(t) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \underbrace{f(x)\, e^{-i\langle t,x\rangle}}_{h(t,x)}\, dx$$

viz důkaz (g), navíc

$|h(t,x)| = |f(x)|$ je integrovatelná majoranta $\Longleftarrow$ $\hat{f}$ je spojitá dle věty o spojitosti podle parametru