Trditev 16 $\quad f,g \in \mathcal{S} \Rightarrow \widehat{f \cdot g} = \hat{f} * \hat{g}$

Γ $\hat{f}, \hat{g} \in \mathcal{S} \subset L^1(\mathbb{R}^d) \Rightarrow \hat{f} * \hat{g} \in L^1(\mathbb{R}^d)$;

$$\widehat{\hat{f} * \hat{g}} = \hat{\hat{f}} \cdot \hat{\hat{g}} = \check{f} \cdot \check{g} = \widecheck{f \cdot g} = \widehat{\widehat{f \cdot g}}$$

$$\Rightarrow \hat{f} * \hat{g} = \widehat{f \cdot g} \ \lrcorner$$

Torej $\mathcal{S}$ je zaprt za konvolucijo:

$$f, g \in \mathcal{S} \Rightarrow f * g = \hat{\check{f}} * \hat{\check{g}} = \widehat{\check{f} \cdot \check{g}} \in \mathcal{S}$$

Izrek 17

① $f \in \mathcal{S} \Rightarrow \|f\|_2 = \|\hat{f}\|_2$

$$\|f\|_2^2 = \int_{\mathbb{R}^d} f \cdot \bar{f} \, dm_d = \int_{\mathbb{R}^d} f \cdot \hat{\check{\bar{f}}} \, dm_d = \int_{\mathbb{R}^d} \hat{f} \, \check{\bar{f}} \, dm_d = \|\hat{f}\|_2^2 \, /$$

ker je $\hat{\check{\bar{f}}}(t) = \int_{\mathbb{R}^d} \check{\bar{f}}(t) e^{-i\langle t, +\rangle} dm_d(t) = \int_{\mathbb{R}^d} \overline{f(-x)} \, e^{-i\langle t, x\rangle} dm_d(x)$

$$= \overline{\int_{\mathbb{R}^d} f(-x) \, e^{-i\langle t, -x\rangle} dm_d(x)} = \overline{\hat{f}}(t).$$

② $\mathcal{S}$ je gost podprostor $L^2(\mathbb{R}^d)$ (ker je $\mathcal{S} \supset \mathcal{D}(\mathbb{R}^d)$),

torej razširitev ex. z Izreka I. 15

③

③ $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \Rightarrow \mathcal{P}(f) = \mathcal{F}(f) \; (= \hat{f})$

- Nechť $r > 0$ a $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $f = 0$ na $\mathbb{R}^d \setminus B(0,r)$

$\Rightarrow$ vlastně $f \in L^2(B(0,r))$

$\Rightarrow$ ex. $\varphi_n \in \mathcal{D}(U(0,r))$: $\varphi_n \to f$ v L^2

Protože $U(0,r)$ má konečnou míru, je i $\varphi_n \to f$ v L^1

Protože $\varphi_n \in \mathcal{D}(U(0,r)) \subset \mathcal{S}(\mathbb{R}^d)$, je $\mathcal{P}(\varphi_n) = \hat{\varphi}_n$

Tedy $\hat{\varphi}_n \rightrightarrows \hat{f}$ (konvergence v $C_0(\mathbb{R})$)

a $\hat{\varphi}_n = \mathcal{P}(\varphi_n) \to \mathcal{P}(f)$ v L^2

ex. podposl., která konverguje s.v., tedy $\mathcal{P}(f) = \hat{f}$ s.v.

$\Rightarrow \mathcal{P}(f) = \hat{f}$ v $L^1 \cap L^2$

- $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ obecná. Označme $f_n := f \cdot \chi_{B(0,n)}$

Dle předchozího bodu je $\mathcal{P}(f_n) = \hat{f_n}$ pro každé $n \in \mathbb{N}$

Máme $f_n \to f$ v $L^2 \Rightarrow \mathcal{P}(f_n) \to \mathcal{P}(f)$ v L^2

$f_n \to f$ v $L^1 \Rightarrow \hat{f_n} \rightrightarrows \hat{f}$

Opět ex. podposl. $\mathcal{P}(f_n) = \hat{f_n}$, která konverguje s.v. k $\mathcal{P}(f)$, tedy $\mathcal{P}(f) = \hat{f}$ s.v., tj. $\mathcal{P}(f) = \hat{f}$ v $L^1 \cap L^2$.