Důkaz Věty 15

H Hilbertův prostor $\quad y \in H \;\ldots\; f_y(x) = \langle x, y \rangle, \; x \in H$

① $f_y \in H^*$: $\quad f_y$ lineární (z definice skal. s.)

$$|f_y(x)| = |\langle x, y\rangle| \leq \|x\| \cdot \|y\| \qquad (C-S)$$

$$\Rightarrow \|f_y\| \leq \|y\|$$

② $\|f_y\| = \|y\|$ $\quad$ – víme $\|f_y\| \leq \|y\|$

$y = 0 \Rightarrow f_0 = 0$

$y \neq 0 \Rightarrow \left\|\frac{y}{\|y\|}\right\| = 1, \quad f_y\left(\frac{y}{\|y\|}\right) = \left\langle \frac{y}{\|y\|}, y \right\rangle = \|y\|$

③ $y \mapsto f_y$ je sdruženě lineární

$$f_{y_1+y_2}(x) = \langle x, y_1 + y_2\rangle = \langle x, y_1\rangle + \langle x, y_2\rangle = f_{y_1}(x) + f_{y_2}(x)$$

$$f_{\lambda y}(x) = \langle x, \lambda y\rangle = \overline{\lambda}\langle x, y\rangle = \overline{\lambda} f_y(x)$$

④ $\forall \varphi \in H^* \; \exists\, y \in H : \quad \varphi = f_y$

- $\varphi = 0 \;\ldots\; y = 0$
- $\varphi \neq 0 \Rightarrow \operatorname{Ker}\varphi$ je uzavřený vlastní podprostor H

$\operatorname{Ker}\varphi \subsetneq H$ uz.

$\Rightarrow (\operatorname{Ker}\varphi)^\perp \neq \{0\} \qquad (\operatorname{Ker}\varphi = ((\operatorname{Ker}\varphi)^\perp)^\perp)$

zvolme $z \in (\operatorname{Ker}\varphi)^\perp, \; \|z\| = 1$

Pak $\forall x \in H$: $\qquad x - \frac{\varphi(x)}{\varphi(z)} \cdot z \in \operatorname{Ker}\varphi$

$$\left[\varphi\left(x - \frac{\varphi(x)}{\varphi(z)} z\right) = \varphi(x) - \frac{\varphi(x)}{\varphi(z)} \cdot \varphi(z) = 0\right]$$

$$\Rightarrow 0 = \left\langle x - \frac{\varphi(x)}{\varphi(z)} z, z \right\rangle = \langle x, z\rangle - \frac{\varphi(x)}{\varphi(z)} \overbrace{\langle z, z\rangle}^{=1}$$

$$\Rightarrow \varphi(x) = \varphi(z) \cdot \langle x, z\rangle = \langle x, \underbrace{\overline{\varphi(z)} \cdot z}_{=y}\rangle$$

Tedy $\varphi = f_{\overline{\varphi(z)}z}$

Důsledek 16 (a) H Hilbert $\Rightarrow H^*$ Hilbert

$\varphi \in H^* \Rightarrow \exists!\, y(\varphi) \in H : \quad \varphi = f_{y(\varphi)}$ | $\varphi \mapsto y(\varphi)$ je sdruženě lineární

definujme $\langle \varphi, \psi \rangle := \langle y(\psi), y(\varphi) \rangle$

Pak
- $\langle \varphi_1 + \varphi_2, \psi \rangle = \langle y(\psi), y(\varphi_1 + \varphi_2) \rangle =$
$= \langle y(\psi), y(\varphi_1) + y(\varphi_2) \rangle = \langle y(\psi), y(\varphi_1) \rangle$
$+ \langle y(\psi), y(\varphi_2) \rangle = \langle \varphi_1, \psi \rangle + \langle \varphi_2, \psi \rangle$
- $\langle \lambda \varphi, \psi \rangle = \langle y(\psi), y(\lambda \varphi) \rangle = \langle y(\psi), \bar{\lambda}\, y(\varphi) \rangle$
$= \lambda \langle y(\psi), y(\varphi) \rangle = \lambda \langle \varphi, \psi \rangle$
- $\langle \psi, \varphi \rangle = \langle y(\varphi), y(\psi) \rangle = \overline{\langle y(\psi), y(\varphi) \rangle} = \langle \varphi, \psi \rangle$
- $\langle \varphi, \varphi \rangle = \langle y(\varphi), y(\varphi) \rangle = \| y(\varphi) \|^2 = \| \varphi \|^2$

b) Hilbertův prostor je reflexivní:

$F \in H^{**} \overset{V15+(a)}{\Longrightarrow} \exists\, \psi \in H^* : \; F(\varphi) = \langle \varphi, \psi \rangle$ pro $\varphi \in H^*$

Pak $y(\psi) \in H$. Ukažme, že $F = \varkappa(y(\psi))$:

$\varphi \in H^* \Rightarrow \varkappa(y(\psi))(\varphi) = \varphi(y(\psi)) = \langle y(\psi), y(\varphi) \rangle$
$= \langle \varphi, \psi \rangle = F(\varphi)$.