Exponenciála a důkaz Věty II. 3

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}, \quad e := \exp(1)$$

Vlastnosti:

(E1) exp je definovaná na $\mathbb{C}$, holomorfní na $\mathbb{C}$ a

$$\exp' z = \exp z, \quad z \in \mathbb{C}$$

⌈ Poloměr konvergence je ∞:

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{1}{n+1} = 0, \quad \text{aplikujme } V1(3)(2).$$

Dle V2(ii) je exp holomorfní na $\mathbb{C}$ a pro $z \in \mathbb{C}$ platí

$$\exp' z = \sum_{n=1}^{\infty} n \cdot \frac{1}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} =$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp z \quad ⌋$$

(E2) $\exp(0) = 1$ ⌈ ihned z definice dosazením ⌋

(E3) $\exp(z+w) = \exp(z) \cdot \exp(w), \quad z, w \in \mathbb{C}$

⌈ Zvolme $w_0 \in \mathbb{C}$ pevné a definujme fci.

$$f(z) = \exp(-z) \cdot \exp(w_0 + z) \Rightarrow f \text{ je holomorfní na } \mathbb{C}$$

$$f'(z) = \exp(-z) \cdot (-1) \cdot \exp(w_0+z) + \exp(-z) \cdot \exp(w_0+z) \cdot 1$$

$$= 0 \quad \text{pro } z \in \mathbb{C}$$

$$\Rightarrow f \text{ je konstantní na } \mathbb{C} \Rightarrow f(z) = f(0) = \exp(0) \cdot \exp(w_0) \underset{(E2)}{=} \exp(w_0)$$

Tedy $\exp(-z)\cdot\exp(w_0+z)=\exp(w_0)$, $z, w_0\in\mathbb{C}$

Aplikujme pro $z=-u$, $w_0=u+v$, $u,v\in\mathbb{C}$,

dostaneme $\exp(u)\cdot\exp(v)=\exp(u+v)$.

To je ono. $\rfloor$

(E4) $\forall z\in\mathbb{C}: \exp(z)\neq 0$

$\ulcorner \exp(z)\cdot\exp(-z)\overset{(E3)}{=}\exp(z-z)=\exp 0\overset{(E2)}{=}1 \lrcorner$

(E5) $\forall z\in\mathbb{C}: \overline{\exp(z)}=\exp(\bar{z})$.

$\ulcorner$ Z definice:

$$\overline{\exp(z)}=\overline{\sum_{n=0}^{\infty}\frac{z^n}{n!}}=\sum_{n=0}^{\infty}\overline{\frac{z^n}{n!}}=\sum_{n=0}^{\infty}\frac{(\bar{z})^n}{n!}=\exp(\bar{z})$$

$z\mapsto\bar{z}$ spojitá, $\overline{a+b}=\bar{a}+\bar{b}$ $\qquad$ $\overline{ab}=\bar{a}\cdot\bar{b}$ $\rfloor$

(E6) $\exp$ zobrazuje $\mathbb{R}$ na $(0,\infty)$, $\exp$ je na $\mathbb{R}$ rostoucí a ryze konvexní

$\ulcorner$ • $z\in\mathbb{R}\Rightarrow\bar{z}=z\Rightarrow\overline{\exp(z)}\overset{(E5)}{=}\exp(\bar{z})=\exp(z)\Rightarrow$ $\exp(z)\in\mathbb{R}$. Tedy $\exp(\mathbb{R})\subset\mathbb{R}$

• $\exp(0)=1$ (E2) & $\exp$ nenabývá 0 (E4) & $\exp$ spojitá (E1) $\Rightarrow$ $\exp(\mathbb{R})$ je interval obsahující 1 a neobsahující 0. Proto $\exp(\mathbb{R})\subset(0,\infty)$

• $\exp z > 0$ pro $z \in \mathbb{R}$, $\exp' = \exp$ (E1) $\Rightarrow$

$\exp' z > 0$ na $\mathbb{R}$, tedy exp je rostoucí na $\mathbb{R}$

$\exp'' z > 0$ na $\mathbb{R}$, tedy exp je ryze konvexní na $\mathbb{R}$

• Protože exp je rostoucí a spojitá na $\mathbb{R}$, k důkazu $\exp(\mathbb{R}) = (0, +\infty)$ stačí ukázat, že

$$\lim_{z \to +\infty} \exp(z) = +\infty, \quad \lim_{z \to -\infty} \exp(z) = 0$$

Limity existují (limita monotónní funkce), tedy

$$\lim_{z \to +\infty} \exp(z) \underset{\uparrow}{=} \lim_{n \to \infty} \exp(n) \overset{(E3)}{=} \lim_{n \to \infty} (\exp(1))^n = +\infty,$$

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protože $\exp(1) \underset{\uparrow}{>} \exp(0) \overset{(E2)}{=} 1$

exp rostoucí

Podobně $\lim_{z \to -\infty} \exp(z) = \lim_{n \to \infty} \exp(-n) = \lim_{n \to \infty} (\exp(-1))^n \underset{\uparrow}{=} 0$

$0 < \exp(-1) < 1$ $\lrcorner$

(E7) $\forall z \in \mathbb{C}: \quad |\exp(z)| = \exp(\operatorname{Re} z)$.

$\ulcorner$ $|\exp(z)|^2 = \exp(z) \cdot \overline{\exp(z)} \overset{(E5)}{=} \exp(z) \cdot \exp(\bar{z}) \overset{(E3)}{=}$

$= \exp(z + \bar{z}) = \exp(2 \operatorname{Re} z) \underset{(E3)}{=} \exp(\operatorname{Re} z)^2$

Dle (E6) je $\exp(\operatorname{Re} z) > 0$, odmocněním máme

$$|\exp(z)| = \exp(\operatorname{Re} z).$$

$\lrcorner$