Written Exam on Mathematics II for IES FSV UK (A)

Summer Semester 2012-2013

Problem 1: Find all the solutions of the system $\mathbb{A}\mathbf{x} = \mathbf{b}$ for the below given matrix $\mathbb{A}$ and given three right-hand side vectors $\mathbf{b}_1$, $\mathbf{b}_2$ a $\mathbf{b}_3$.

$$\mathbb{A} = \begin{pmatrix} 2 & 1 & 2 & 5 \\ 1 & 2 & 2 & 5 \\ 5 & 1 & 2 & 2 \\ 5 & 2 & 2 & 1 \end{pmatrix}, \mathbf{b}_1 = \begin{pmatrix} -3 \\ -4 \\ 3 \\ 4 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

(10 points)

Problem 2: Determine and draw the domain of the function

$$f(x, y) = \sqrt{x(y - \sin x)},$$

compute its partial derivatives with respect to all the variables at all points where they exist.

(10 points)

Problem 3: Let us consider the equation

$$\operatorname{arctg}(2x + y) + \sin(y^2 - x^2) + \frac{\pi}{4} = 0$$

and the point $[-1, 1]$. Show that this equation defines a C^∞ function $y = f(x)$ defined on a neighborhood of -1, which satisfies $f(-1) = 1$. Compute $f'(-1)$, $f''(-1)$ and determine the equation of the tangent line to the graph of f at the point $[-1, f(-1)]$. (10 points)

Problem 4: Determine sup and inf of the function f on the set M and decide whether these values are attained, if

$$f(x, y, z) = xz \text{ and } M = \{[x, y, z] \in \mathbb{R}^3 : x^2 + 2y^2 \leq 3, z^2 - y^2 = 1\}$$

(15 points)

Problem 5: Compute the following antiderivative on maximal possible intervals:

$$\int \frac{7x^4}{(x^2 - 1)(x^2 + 2x + 4)} \, \mathrm{d}x$$

(15 points)

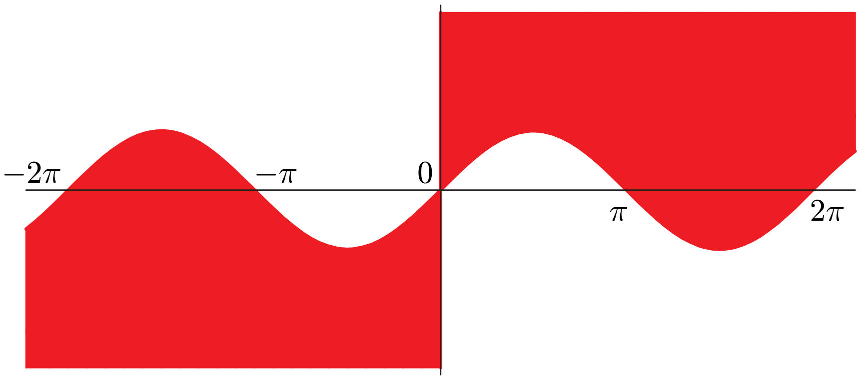
Answers to the Written Exam on Mathematics II for IES FSV UK (A)

Summer Semester 2012-2013

Problem 1: For $\mathbf{b}_1$: infinitely many solutions of the form $[t+2, t+1, -4t-4, t]$, $t \in \mathbf{R}$. For $\mathbf{b}_2$: infinitely many solutions of the form $[t, t, \frac{1}{2} - 4t, t]$, $t \in \mathbf{R}$. For $\mathbf{b}_3$: no solution.

Problem 2: $D_f = \{[x,y] \in \mathbf{R}^2 : x \geq 0 \ \& \ y \geq \sin x \text{ or } x \leq 0 \ \& \ y \leq \sin x\}$.

Picture of the domain:



$\frac{\partial f}{\partial x}(x,y) = \frac{1}{2\sqrt{x(y-\sin x)}} \cdot (y - \sin x - x\cos x)$ and $\frac{\partial f}{\partial y}(x,y) = \frac{1}{2\sqrt{x(y-\sin x)}} \cdot x$; both partial derivatives on the set $\{[x,y] \in \mathbf{R}^2 : x > 0 \ \& \ y > \sin x \text{ or } x < 0 \ \& \ y < \sin x\}$. At points $[0,y]$, $y \in \mathbf{R}$, the partial derivative with respect to x has no sense (as no horizontal segment centered at $[0,y]$ belongs to D_f) and the partial derivative with respect to y is zero. At points $[x, \sin x]$, $x \in \mathbf{R} \setminus \{0\}$ the partial derivative with respect to y has no sense (as no vertical segment centered at $[x, \sin x]$ is contained in D_f). Partial derivative with respect to x needs to be computed only in points $[\frac{\pi}{2} + 2k\pi, 1]$, $k \in \mathbf{Z}$, $k \geq 0$ and in points $[-\frac{\pi}{2} + 2k\pi, -1]$, $k \in \mathbf{Z}$, $k \leq 0$ (for other points there is no horizonotal segment with the respective center contained in D_f). The computation shows that these partial derivatives do not exist (since the respective one-sided limits are different).

Problem 3: $f'(-1) = -\frac{6}{5}$, $f''(-1) = -\frac{12}{25}$, tangent line $y = 1 - \frac{6}{5}(x+1)$.

Problem 4: Maximum $\frac{5}{2\sqrt{2}}$ at points $[\sqrt{\frac{5}{2}}, \frac{1}{2}, \frac{\sqrt{5}}{2}]$, $[\sqrt{\frac{5}{2}}, -\frac{1}{2}, \frac{\sqrt{5}}{2}, \frac{3}{2}]$, $[-\sqrt{\frac{5}{2}}, \frac{1}{2}, -\frac{\sqrt{5}}{2}]$, $[-\sqrt{\frac{5}{2}}, -\frac{1}{2}, -\frac{\sqrt{5}}{2}]$; minimum $-\frac{5}{2\sqrt{2}}$ at points $[-\sqrt{\frac{5}{2}}, \frac{1}{2}, \frac{\sqrt{5}}{2}]$, $[-\sqrt{\frac{5}{2}}, -\frac{1}{2}, \frac{\sqrt{5}}{2}]$, $[\sqrt{\frac{5}{2}}, \frac{1}{2}, -\frac{\sqrt{5}}{2}]$, $[\sqrt{\frac{5}{2}}, -\frac{1}{2}, -\frac{\sqrt{5}}{2}]$.

Problem 5: $\int \frac{7x^4}{(x^2-1)(x^2+2x+4)} \, \mathrm{d}x \stackrel{c}{=} 7x + \frac{1}{2}\log|x-1| - \frac{7}{6}\log|x+1| - \frac{20}{3}\log(x^2+2x+4) - \frac{8}{\sqrt{3}} \operatorname{arctg} \frac{x+1}{\sqrt{3}}$ on each of the three intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, +\infty)$.