~~16/12/05~~ | MATRIX 1 | PROBLEM C1

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 0 & 1 & 0 & 0 \\ 1 & 3 & 9 & 27 & 0 & 0 & 1 & 0 \\ 1 & 4 & 16 & 64 & 0 & 0 & 0 & 1 \end{array}\right) \sim \begin{matrix} \\ \\ -2 \\ -3 \end{matrix} \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 7 & -1 & 1 & 0 & 0 \\ 0 & 2 & 8 & 26 & -1 & 0 & 1 & 0 \\ 0 & 3 & 15 & 63 & -1 & 0 & 0 & 1 \end{array}\right)$$

1pt

$$\sim \begin{matrix} \\ \\ \\ -3. \end{matrix} \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 7 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 12 & 1 & -2 & 1 & 0 \\ 0 & 0 & 6 & 42 & 2 & -3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 7 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 12 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 6 & -1 & 3 & -3 & 1 \end{array}\right)$$

1pt

$$\sim \begin{matrix} \\ -7 \\ \\ 6. \end{matrix} \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 7 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 6 & \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \end{array}\right) \sim \begin{matrix} -1 \\ \\ -3 \\ \\ \end{matrix} \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & \frac{7}{6} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{6} \\ 0 & 1 & 3 & 0 & \frac{1}{6} & -\frac{5}{2} & \frac{7}{2} & -\frac{7}{6} \\ 0 & 0 & 1 & 0 & \frac{3}{2} & -4 & \frac{7}{2} & -1 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \end{array}\right)$$

$$\sim \begin{matrix} -1 \\ \\ \\ \\ \end{matrix} \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & -\frac{1}{3} & \frac{7}{2} & -3 & 5/6 \\ 0 & 1 & 0 & 0 & -\frac{13}{3} & \frac{19}{2} & -7 & \frac{11}{6} \\ 0 & 0 & 1 & 0 & 3/2 & -4 & 7/2 & -1 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 4 & -6 & 4 & -1 \\ 0 & 1 & 0 & 0 & -13/3 & \frac{19}{2} & -7 & \frac{11}{6} \\ 0 & 0 & 1 & 0 & 3/2 & -4 & 7/2 & -1 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \end{array}\right)$$

A^{-1} 1pt

$$\frac{1}{6} - \frac{9}{2} = \frac{1}{6} - \frac{27}{6} = -\frac{26}{6} = -\frac{13}{3}$$

$$B^{-1} = \begin{pmatrix} \frac{4}{3} & \frac{2}{5} & -6 & -1 \\ -\frac{7}{3} & -\frac{13}{30} & \frac{19}{2} & \frac{11}{6} \\ \frac{7}{6} & \frac{3}{20} & -4 & -1 \\ -\frac{1}{6} & -\frac{1}{60} & 1/2 & 1/6 \end{pmatrix}$$

1^{st} colum $= \frac{1}{3}$ of 3^{rd} colum of A^{-1}
2^{nd} colum $= \frac{1}{10}$ of 1^{st} — " —
3^{rd} colum $=$ 2^{nd} — " —
4^{th} colum $=$ 4^{th} — " —

3pts

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$\boxed{\text{PROBLEM C2}}$

PD2 $f(x,y) = \sqrt{y^2-4x^2}$

D_f: $y^2 - 4x^2 \geq 0$

$(y-2x)(y+2x) \geq 0$

$y \geq 2x$ & $y \geq -2x$ $\quad -y \leq 2x \leq y$

or

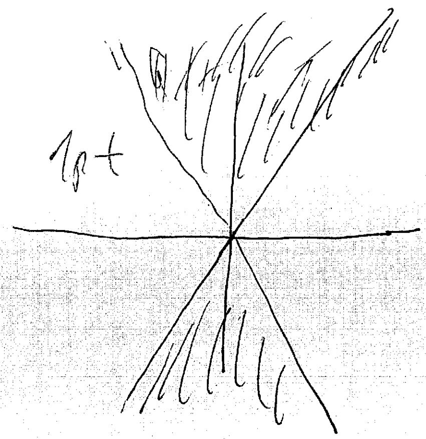
$y \leq 2x$ & $y \leq -2x$ $\quad -y \geq 2x \geq y$

2pts

Hence $D_f = \{[x,y] : y \geq 0$ & $-\frac{y}{2} \leq x \leq \frac{y}{2}$

or

$y \leq 0$ & $\frac{y}{2} \leq x \leq -\frac{y}{2}\}$



1pt

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{y^2-4x^2}} \cdot (-8x)$$

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y^2-4x^2}} \cdot 2y$$

2pts

if $y>0$ & $-\frac{y}{2}<x<\frac{y}{2}$

or

$y<0$ & $\frac{y}{2}<x<-\frac{y}{2}$

1pt

Cutbreaking points: $x = y/2$ or $x = -y/2$

The only p.d. to compute is $\frac{\partial f}{\partial y}(0,0)$. The remaining points: no segment in

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{\sqrt{h^2}-0}{h} = \lim_{h\to 0} \frac{|h|}{h} \quad \text{.. does not exist.}$$

1pt

at $x = y/2$, $x = -y/2$, $\ne 0$ ... no p.d. has a sense (neither vertical nor horizontal segment in D_f) 2pts

at $[0,0]$: $\frac{\partial f}{\partial x}$ no sense (no horizontal segment in D_f) 1pt

$\frac{\partial f}{\partial y}(0,0)$

IFTS ~~[illegible]~~

PROBLEM C3

4

$\underbrace{\arcsin(x+y) + \sin(x+y^2)}_{F(x,y)} = 0 \quad , \quad [-1,1]$

3pts

① $F \in C^\infty(\{(x,y) \in \mathbb{R}^2 : -1 < x+y < 1\})$

$[-1,1] \in ?$

② $F(-1,1) = \arcsin(-1+1) + \sin(-1+1) = 0$

③ $\frac{\partial F}{\partial y}(-1,1) = \left(\frac{1}{\sqrt{1-(x+y)^2}} + \cos(x+y^2) \cdot 2y \right)_{\substack{x=-1 \\ y=1}} = 1 + 1 \cdot 2 = 3 \neq 0$

$\Rightarrow$ there exists a C^∞ function f with the required properties,

1pt in part. $\arcsin(x+f(x)) + \sin(x+f(x)^2) = 0$ on a nbhood of -1

differentiate: $\frac{1}{\sqrt{1-(x+f(x))^2}}(1+f'(x)) + \cos(x+f(x)^2) \cdot (1+2f(x)f'(x)) = 0$ 1 pt

$\left.\begin{matrix} x=-1 \\ f(-1)=1 \end{matrix}\right.$: $\frac{1}{\sqrt{1}}(1+f'(-1)) + \cos 0 \; (1+2f'(-1)) = 0$

$3f'(-1) + 2 = 0 \Rightarrow f'(-1) = -\frac{2}{3}$ 1pt

tangent line: $y = 1 - \frac{2}{3}(x+1)$ 1pt

second derivative:

$-\frac{1}{2} \cdot (1-(x+f(x))^2)^{-\frac{3}{2}} \cdot (-2)(x+f(x))(1+f'(x))^2 + \frac{1}{\sqrt{1-(x+f(x))^2}} \cdot f''(x)$

2pts $- \sin(x+f(x)^2)(1+2f(x)f'(x))^2 + \cos(x+f(x)^2)(2f'(x)f'(x) + 2f(x)f''(x)) = 0$

$\left.\begin{matrix} x=-1 \\ f(-1)=1 \\ f'(-1)=-\frac{2}{3} \end{matrix}\right\}$: $-\frac{1}{2} \cdot 1 \cdot (-2) \cdot 0 \, (\dots)^2 + 1 \cdot f''(-1)$

$- \sin 0 \cdot (\dots) + \cos 0 \cdot (2 \cdot \frac{4}{9} + 2f''(-1)) = 0$

$3f''(-1) + \frac{8}{9} = 0$

$f''(-1) = -\frac{8}{27}$ 1pt

PROBLEM C4

$f(x,y,z) = x - y$

$M = \{[x,y,z] \in \mathbb{R}^3 : x^2 + 2y^2 + z^2 = 4,\ xy + \sqrt{\tfrac{3}{2}} \geq 0\}$

① Existence of extrema: • f is cts on $\mathbb{R}^3$

2pts

• M is closed ("=" and "≥")

and bounded $[x,y,z] \in M \Rightarrow x^2+y^2+z^2 \leq 4 \Rightarrow$ M is contained in the closed ball centered at 0 with radius 2

$\Rightarrow$ M is compact

So extrema do exist provided $M \neq \emptyset$

② $M = M_1 \cup M_2$: $M_1 = \{[x,y,z] \in \mathbb{R}^3 : x^2+2y^2+z^2 = 4,\ xy + \sqrt{\tfrac{3}{2}} > 0\}$

1pt $M_2 = \{[x,y,z] \in \mathbb{R}^3 : x^2+2y^2+z^2 = 4,\ xy + \sqrt{\tfrac{3}{2}} = 0\}$

For future use: $Df = [1,-1,0]$, $g_1(x,y,z) = x^2+2y^2+z^2-4$

$Dg_1 = [2x, 4y, 2z]$

$g_2(x,y,z) = xy + \sqrt{\tfrac{3}{2}}$, $Dg_2 = [y, x, 0]$

③ On M_1: At points of extrema

1pt either • $Dg_1 = 0 \Leftrightarrow x = y = z = 0$, but $[0,0,0] \notin M_1$ not possible

or $\exists \lambda \in \mathbb{R}$: $Df + \lambda Dg_1 = 0$

$1 + \lambda 2x = 0$

$-1 + \lambda \cdot 4y = 0$

$\lambda \cdot 2z = 0$

1pt

$\lambda = 0$ or $z = 0$. But $\lambda = 0 \Rightarrow 1 = 0$ impossible

So $z = 0$

Further: $2y + x = 0 \Rightarrow x = -2y$

So, $4y^2 + 2y^2 = 4 \Rightarrow y^2 = \tfrac{2}{3}$ $y = \pm\sqrt{\tfrac{2}{3}}$ 2pts

$[-2\sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{2}{3}}, 0]$

$[2\sqrt{\tfrac{2}{3}}, -\sqrt{\tfrac{2}{3}}, 0]$

$xy + \sqrt{\tfrac{3}{2}} = -\tfrac{4}{3} + \sqrt{\tfrac{3}{2}} \overset{?}{>} 0$

$\sqrt{\tfrac{3}{2}} \overset{?}{>} \tfrac{4}{3}$ $\tfrac{3}{2} > \tfrac{16}{9}$ $27 > 32$ NO

$\Rightarrow$ no points in M_1

(4) On M_2:

Case 1: ∇g_1 and ∇g_2 linearly dependent:

~~$\nabla g_2 = \alpha \cdot \nabla g_1$: $2x = \alpha y$ $y =$~~

2pts

$\nabla g_1 = \alpha \cdot \nabla g_2$: $\left.\begin{array}{l} 2x = \alpha y \\ 4y = \alpha x \\ 2z = \alpha \cdot 0 \end{array}\right) \Rightarrow z = 0$ and $2x^2 - 4y^2 = 0$

$x^2 = 2y^2$

$2y^2 + 2y^2 + 0^2 = 4$

$\Rightarrow y^2 = 1$

$x^2 = 2$

then $xy = \pm\sqrt{2} \neq \sqrt{\frac{3}{2}}$ ... impossible

$\nabla g_2 = \alpha \cdot \nabla g_1$: $\alpha \neq 0$ impossible by the above (then $\nabla g_1 = \frac{1}{\alpha} \nabla g_2$)

$\alpha = 0$: $x = 0, y = 0$ ... impossible since $xy = -\sqrt{\frac{3}{2}}$

Case 2: $\nabla f + \lambda \nabla g_1 + \mu \nabla g_2 = 0$

1pt

$1 + \lambda \cdot 2x + \mu \cdot y = 0$

$-1 + \lambda \cdot 4y + \mu \cdot x = 0$

$\lambda \cdot 2z = 0$

$z = 0$ or $\lambda = 0$

$\lambda = 0$: $\left.\begin{array}{l} 1 + \mu y = 0 \\ -1 + \mu x = 0 \end{array}\right\} x + y = 0$

$y = -x$

$-x^2 + \sqrt{\frac{3}{2}} = 0$

$x^2 = \sqrt{\frac{3}{2}}$

$y^2 = \sqrt{\frac{3}{2}}$

$\sqrt{\frac{3}{2}} + 2\sqrt{\frac{3}{2}} + z^2 = 4$

$z^2 = 4 - 3 \cdot \sqrt{\frac{3}{2}}$

$\left(4 - 3\sqrt{\frac{3}{2}} > 0 \text{ as } 16 > 9 \cdot \frac{3}{2} \Leftrightarrow 32 > 27\right)$

if $z = 0$, then: $x^2 + 2y^2 = 4$

$xy = -\sqrt{\frac{3}{2}}$

$y = -\sqrt{\frac{3}{2}} \cdot \frac{1}{x}$

$y^2 = \frac{3}{2} \cdot \frac{1}{x^2}$

$x^2 + \frac{3}{x^2} = 4$

$x^4 - 4x^2 + 3 = 0$

$(x^2 - 1)(x^2 - 3) = 0$

$[1, -\sqrt{\frac{3}{2}}, 0] \xmapsto{f} 1 + \sqrt{\frac{3}{2}}$

$[-1, \sqrt{\frac{3}{2}}, 0] \longmapsto -1 - \sqrt{\frac{3}{2}}$

$[\sqrt{3}, -\frac{1}{\sqrt{2}}, 0] \longmapsto \sqrt{3} + \frac{1}{\sqrt{2}}$

$[-\sqrt{3}, \frac{1}{\sqrt{2}}, 0] \longmapsto -\sqrt{3} - \frac{1}{\sqrt{2}}$

3pts

Candidates of $\lambda = 0$:

$$\left[\sqrt[4]{\tfrac{3}{2}},\ -\sqrt[4]{\tfrac{3}{2}},\ \pm\sqrt{4-3\sqrt{\tfrac{3}{2}}}\right] \xrightarrow{f} 2\sqrt[4]{\tfrac{3}{2}}$$

$$\left[-\sqrt[4]{\tfrac{3}{2}},\ \sqrt[4]{\tfrac{3}{2}},\ \pm\sqrt{4-3\sqrt{3/2}}\right] \xrightarrow{f} -2\sqrt[4]{\tfrac{3}{2}}$$

Comparison:

$$2\sqrt[4]{\tfrac{3}{2}} \overset{?}{>} 1+\sqrt{\tfrac{3}{2}}$$
$$4\cdot\sqrt{\tfrac{3}{2}} > 1 + 2\sqrt{\tfrac{3}{2}} + \tfrac{3}{2}$$
$$2\sqrt{\tfrac{3}{2}} > \tfrac{5}{2}$$
$$\sqrt{\tfrac{3}{2}} > \tfrac{5}{4}$$
$$\tfrac{3}{2} > \tfrac{25}{16}$$
$$48 > 50 \quad \text{NO}$$
$$\Rightarrow 1+\sqrt{\tfrac{3}{2}} > 2\cdot\sqrt[4]{\tfrac{3}{2}}$$

$$1+\sqrt{\tfrac{3}{2}} \overset{?}{>} \sqrt{3} + \tfrac{1}{\sqrt{2}}$$
$$\sqrt{2}+\sqrt{3} > \sqrt{6} + 1$$
$$2 + 2\sqrt{6} + 3 > 6 + 2\sqrt{6} + 1$$
$$5 > 7$$
NO
$$\Rightarrow \sqrt{3} + \tfrac{1}{\sqrt{2}} > 1 + \sqrt{\tfrac{3}{2}}$$

2 pts

So, maximum $\sqrt{3}+\frac{1}{\sqrt{2}}$ at $\left[\sqrt{3}, -\frac{1}{\sqrt{2}}, 0\right]$

minimum $-\sqrt{3}-\frac{1}{\sqrt{2}}$ at $\left[-\sqrt{3}, \frac{1}{\sqrt{2}}, 0\right]$

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PROBLEM C5

$$\int \frac{5x^4}{(x^2-2x+2)(x^2-x-2)}\,dx$$

① division

$$(x^2-2x+2)(x^2-x-2) = x^4 - x^3 - 2x^2 - 2x^3 + 2x^2 + 4x + 2x^2 - 2x - 4$$

$$= x^4 - 3x^3 + 2x^2 + 2x - 4$$

$$5x^4 : (x^4 - 3x^3 + 2x^2 + 2x - 4) = 5$$

$$15x^3 - 10x^2 - 10x + 20$$

$$\frac{5x^4}{(x^2-2x+2)(x^2-x-2)} = 5 + \frac{15x^3 - 10x^2 - 10x + 20}{(x^2-2x+2)(x^2-x-2)}$$ 1pt

② partial fractions:

$$\frac{15x^3 - 10x^2 - 10x + 20}{(x^2-2x+2)(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{Cx+D}{x^2-2x+2}$$ 1pt

$$15x^3 - 10x^2 - 10x + 20 = A(x+1)(x^2-2x+2) + B(x-2)(x^2-2x+2) + (Cx+D)(x-2)(x+1)$$

$x = -1$:

$$-15 - 10 + 10 + 20 = B \cdot (-3) \cdot 5$$

$$5 = -15B \Rightarrow B = -\frac{1}{3}$$

$x = 2$:

$$\underset{120}{15 \cdot 8} - 40 - \cancel{20} + \cancel{20} = A \cdot 3 \cdot (4 - 4 + 2)$$

$$80 = \cancel{24}\,6A$$

$$A = \frac{40}{3}$$

at x^3: $15 = A + B + C = \frac{40}{3} - \frac{1}{3} + C = 13 + C$

$\cancel{C = 15 - \frac{73}{21}}$

$\Rightarrow C = 2$

at x^0: $20 = 2A - 4B - 2D = \frac{80}{3} + \frac{4}{3} - 2D = 28 - 2D$

$\Rightarrow 2D = 8 \Rightarrow D = 4$

3pts

(3) $$\int \frac{2x+4}{x^2-2x+2}\,dx = \int \frac{(2x-2)}{x^2-2x+2} + \frac{6}{x^2-2x+2}\,dx =$$

1pt

$$= \log(x^2-2x+2) + 6\int \frac{1}{x^2-2x+2}\,dx$$

1pt

3pts

$$= 6\int \frac{1}{(x-1)^2+1}\,dx \overset{c}{=} 6\arctg(x-1)$$

2pts

<u>The result:</u>

$$\int \frac{5x^4}{(x^2-2x+2)(x^2-x-2)}\,dx \overset{c}{=} 5x + \frac{40}{3}\lg|x-2| - \frac{1}{3}\lg|x+1| + \lg(x^2-2x+2) + 6\arctg(x-1)$$

on $(-\infty,-1)$, $(-1,2)$ and $(2,+\infty)$ 2pts

1pt